Linear Algebra and its Applications 469 (2015) 539–568

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Multiplicative Lidskii's inequalities and optimal perturbations of frames ☆

Pedro G. Massey [a,b], Mariano A. Ruiz [a,b,*], Demetrio Stojanoff [a,b]

[a] Depto. de Matemática, FCE-UNLP, La Plata, Argentina
[b] IAM-CONICET, Argentina

ARTICLE INFO

Article history:
Received 29 August 2014
Accepted 5 December 2014
Available online 29 December 2014
Submitted by R. Brualdi

MSC:
42C15
15A60

Keywords:
Frames
Perturbation of frames
Majorization
Lidskii's inequality
Convex potentials

ABSTRACT

In this paper we study two design problems in frame theory: on the one hand, given a fixed finite frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ for $\mathbb{C}^d$ we compute those dual frames $\mathcal{G}$ of $\mathcal{F}$ that are optimal perturbations of the canonical dual frame for $\mathcal{F}$ under certain restrictions on the norms of the elements of $\mathcal{G}$. On the other hand, we compute those $V \cdot \mathcal{F} = \{Vf_j\}_{j\in\mathbb{I}_n}$ – for invertible operators V which are close to the identity – that are optimal perturbations of $\mathcal{F}$. That is, we compute the optimal perturbations of $\mathcal{F}$ among frames $\mathcal{G} = \{g_j\}_{j\in\mathbb{I}_n}$ that have the same linear relations as $\mathcal{F}$. In both cases, optimality is measured with respect to submajorization of the eigenvalues of the frame operators. Hence, our optimal designs are minimizers of a family of convex potentials that include the frame potential and the mean squared error. The key tool for these results is a multiplicative analogue of Lidskii's inequality in terms of log-majorization and a characterization of the case of equality.

☆ Partially supported by CONICET (PIP 0435/10) and Universidad Nacional de La Plata (UNLP 11X681).

* Corresponding author.
E-mail addresses: massey@mate.unlp.edu.ar (P.G. Massey), mruiz@mate.unlp.edu.ar (M.A. Ruiz), demetrio@mate.unlp.edu.ar (D. Stojanoff).

http://dx.doi.org/10.1016/j.laa.2014.12.004
0024-3795/

1. Introduction

A finite frame for $\mathbb{C}^d$ is a sequence $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ that spans $\mathbb{C}^d$, where $\mathbb{I}_n = \{1, \ldots, n\}$ (for a detailed exposition on frames and several recent research topics within this theory see [8,9] and the references therein). Given a frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$, a sequence $\mathcal{G} = \{g_j\}_{j\in\mathbb{I}_n}$ is called a dual frame for $\mathcal{F}$ if for every $f \in \mathbb{C}^d$ the following reconstruction formulas hold:

$$f = \sum_{j\in\mathbb{I}_n} \langle f, g_j\rangle f_j \quad \text{and} \quad f = \sum_{j\in\mathbb{I}_n} \langle f, f_j\rangle g_j.$$

Hence, frames provide a (possibly redundant) linear-encoding scheme for vectors in $\mathbb{C}^d$.

Let $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ be a frame for $\mathbb{C}^d$ and let $\mathcal{D}(\mathcal{F})$ denote the set of dual frames for $\mathcal{F}$. There is a distinguished dual called the canonical dual of $\mathcal{F}$, denoted $\mathcal{F}^{\#} \in \mathcal{D}(\mathcal{F})$, which is a natural choice in several ways. But in case $n > d$ it is well known that $\mathcal{D}(\mathcal{F})$ has a rich structure (this last fact is one of the main advantages of frames over bases $\mathcal{B} = \{v_j\}_{i\in\mathbb{I}_d}$ for which $\mathcal{D}(\mathcal{B})$ becomes a singleton). Thus, in applied situations, the structure of $\mathcal{D}(\mathcal{F})$ can be exploited to obtain numerically stable encoding–decoding schemes derived from the dual pair $(\mathcal{F}, \mathcal{G})$, for some choice of dual frame $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ beyond $\mathcal{F}^{\#}$; this is the starting point of the so-called (optimal) design problems for dual frames (see [13,15,17–19]).

Another research topic in frame theory is the design of (optimal) stable configurations of vectors (frames) under certain restrictions. Typically, the stability of a frame $\mathcal{F}$ is measured in terms of the spread the eigenvalues of the positive semidefinite operator $S_{\mathcal{F}} = \sum_{j\in\mathbb{I}_n} f_j \otimes f_j$. One of the most important examples of such a measure is the frame potential of $\mathcal{F}$, denoted by $\mathrm{FP}(\mathcal{F})$, introduced in [5]; explicitly, for a sequence $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ then

$$\mathrm{FP}(\mathcal{F}) = \sum_{j,k\in\mathbb{I}_n} \left|\langle f_j, f_k\rangle\right|^2 = \mathrm{tr}\left(S_{\mathcal{F}}^2\right).$$

In [5,7] it is shown that minimizers of the frame potential, within convenient sets of frames, have many nice structural features and are optimal in several ways. Recently, there has also been interest in the so-called mean squared error of $\mathcal{F}$, denoted $\mathrm{MSE}(\mathcal{F})$, given by $\mathrm{MSE}(\mathcal{F}) = \mathrm{tr}(S_{\mathcal{F}}^{-1})$ (see [11,16,20]).

It turns out that there is a structural measure of optimality, called sub-majorization, that allows to deal with both the frame potential and the mean squared error. This pre-order relation, defined between eigenvalues of frame operators, has proved useful in explaining the structure of minimizers of convex potentials (see [16]). Sub-majorization has also been useful in obtaining the structure of optimal vector configurations as well (see [19,21]). In turn, sub-majorization relations imply a family of tracial inequalities in terms of convex functions, that contain the frame potential and mean squared errors. We point out that these tracial inequalities have interest in their own right and collectively characterize sub-majorization.

In this paper we consider two optimal design problems, where optimality is measured in terms of sub-majorization. On the one hand, given a fixed frame $\mathcal{F}$, we consider the problem of designing a dual frame $\mathcal{G} = \{g_j\}_{j\in\mathbb{I}_n} \in \mathcal{D}(\mathcal{F})$ such that $\sum_{j\in\mathbb{I}_n} \|g_j\|^2 \geq t$ for some appropriate $t > 0$, the distance between $\mathcal{G}$ and $\mathcal{F}^{\#}$ is controlled by some $\epsilon > 0$ and such that the spread of the eigenvalues of $S_{\mathcal{G}}$ is minimal (with respect to sub-majorization) among the eigenvalues of $S_{\mathcal{G}'}$ for all such $\mathcal{G}'$. Thus, we refine the analysis of the optimal design problem for dual frames obtained in [19] (see Section 3.1 for a detailed description of this problem and some further motivations). Since we keep control of the distance of these frames to the canonical dual, we consider this optimal solution as a perturbation of the canonical dual (that improves some of its numerical features).

On the other hand, given a fixed frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ for $\mathbb{C}^d$ with $n > d$ we consider the problem of designing an invertible operator V acting on $\mathbb{C}^d$ which is close the identity operator and such that, if we denote by $V \cdot \mathcal{F} = \{V f_j\}_{j\in\mathbb{I}_n}$ then, the spread of the eigenvalues of $S_{V\cdot\mathcal{F}}$ is minimal (with respect to sub-majorization) among the eigenvalues of $S_{V'\cdot\mathcal{F}}$ for all such V' (see Section 3.2 for a detailed description of this problem and further motivations). Notice that a frame $\mathcal{G}$ for $\mathbb{C}^d$ can be written as $V \cdot \mathcal{F}$ for some invertible operator V acting on $\mathbb{C}^d$ if and only if the frames $\mathcal{F}$ and $\mathcal{G}$ have the same linear relations. Two such frames are called equivalent (see [2]): hence, we search for equivalent (in the previous sense) frames $V \cdot \mathcal{F}$ that are close to $\mathcal{F}$ and such that they improve some numerical features of $\mathcal{F}$.

In order to tackle both problems above, we introduce abstract models for them within the framework of matrix analysis. Although the frame problems seem unrelated, it turns out that the abstract model for the design of optimal duals plays a crucial role in the analysis of the abstract model for the perturbations by equivalent frames.

The key tools for these results are the multiplicative analogue of Lidskii's inequality in terms of log-majorization obtained by Li and Mathias in [14], and a characterization of the case of equality (see Appendix A). We also use the optimality results proved in [20], which are based in the additive case for Lidskii's inequality and the case of equality, studied in the appendix of that paper.

The paper is organized as follows. In Section 2, after setting the general notation used throughout the paper, we describe the basic framework of finite frames that we shall need, together with a brief description of general convex potentials. We also include a description of sub-majorization and log-majorization which are two notions from matrix theory. In Section 3 we give a detailed description of the two problems in frame theory that we consider in this note, using the notation and terminology from Section 2. In Section 4 we first introduce an abstract model for the design of optimal dual frames with restrictions, and apply tools from matrix analysis to obtain optimality results; we then apply these results to the original frame problem. Similarly, in Section 5 we first analyze an abstract model for the design of optimal perturbations of a frame by equivalent frames and then apply the results of the abstract model to the original frame problem. Finally, in Appendix A we develop some aspects of the multiplicative Lidskii's inequality with respect to log-majorization which are needed for the analysis of the abstract model in Section 5.

2. Preliminaries

2.1. General notation

Throughout this work we shall use the following notation: the space of complex $d \times d$ matrices is denoted by $\mathcal{M}_d(\mathbb{C})$ and $\mathcal{M}_d(\mathbb{C})^+$ the set of positive semidefinite matrices. $\mathcal{G}l(d)$ is the group of invertible elements of $\mathcal{M}_d(\mathbb{C})$ and $\mathcal{G}l(d)^+ = \mathcal{M}_d(\mathbb{C})^+ \cap \mathcal{G}l(d)$. If $T \in \mathcal{M}_d(\mathbb{C})$, we denote by $\|T\|$ its spectral norm, by $\operatorname{rk} T = \dim R(T)$ the rank of T, and by $\operatorname{tr} T$ the trace of T.

If $W \subseteq \mathbb{C}^d$ is a subspace we denote by $P_W \in \mathcal{M}_d(\mathbb{C})^+$ the orthogonal projection onto W. Given $x, y \in \mathbb{C}^d$ we denote by $x \otimes y \in \mathcal{M}_d(\mathbb{C})$ the rank one matrix given by

$$x \otimes y(z) = \langle z, y\rangle x \quad \text{for every } z \in \mathbb{C}^d. \tag{1}$$

Note that if $\|x\| = 1$ then $x \otimes x = P_{\operatorname{span}\{x\}}$.

Given $d \in \mathbb{N}$ we denote by $\mathbb{I}_d = \{1, \ldots, d\} \subseteq \mathbb{N}$ and $\mathbb{1} = \mathbb{1}_m \in \mathbb{R}^m$ denotes the vector with all its entries equal to 1. For vectors $x, y \in \mathbb{R}^m$ we denote

$$x \circ y = (x_1 y_1, \ldots, x_m y_m), \qquad \operatorname{tr} x = \sum_{i \in \mathbb{I}_d} x_i,$$

and $x^\downarrow$ (resp. $x^\uparrow$) the rearrangement of x in decreasing (resp. increasing) order. We denote by $(\mathbb{R}^d)^\downarrow = \{x \in \mathbb{R}^d : x = x^\downarrow\}$ the set of downwards ordered vectors, and $(\mathbb{R}^d)^\uparrow = \{x \in \mathbb{R}^d : x = x^\uparrow\}$.

Given a positive semidefinite matrix $S \in \mathcal{M}_d(\mathbb{C})$, we write $\lambda(S) = \lambda^\downarrow(S) \in (\mathbb{R}^d)^\downarrow$ the vector of eigenvalues of S – counting multiplicities – arranged in decreasing order. Similarly we denote by $\lambda^\uparrow(S) \in (\mathbb{R}^d)^\uparrow$ the reverse ordered vector of eigenvalues of S.

2.2. Basic framework of finite frames

Let $d, n \in \mathbb{N}$, with $d \leq n$. A family $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ of vectors in $\mathbb{C}^d$ is a frame for $\mathbb{C}^d$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{i \in \mathbb{I}_n} \left|\langle x, f_i\rangle\right|^2 \leq B\|x\|^2 \quad \text{for every } x \in \mathbb{C}^d. \tag{2}$$

The optimal frame bounds, denoted by $A_{\mathcal{F}}, B_{\mathcal{F}}$ are the optimal constants in Eq. (2). If $A_{\mathcal{F}} = B_{\mathcal{F}}$ we call $\mathcal{F}$ a tight frame. For finite frames, it is clear that (2) is equivalent to $\operatorname{span}\{f_i : i \in \mathbb{I}_n\} = \mathbb{C}^d$.

Given a family $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ of vectors in $\mathbb{C}^d$, we consider its analysis operator $T_\mathcal{F} : \mathbb{C}^d \to \mathbb{C}^n$, given by

$$T_\mathcal{F} x = \big(\langle x, f_i\rangle\big)_{i\in\mathbb{I}_n}, \quad \text{for every } x \in \mathbb{C}^d. \tag{3}$$

Its adjoint $T^*_\mathcal{F} : \mathbb{C}^n \to \mathbb{C}^d$ is called the synthesis operator and it is given by $T^*_\mathcal{F}\mathbf{a} = \sum_{i\in\mathbb{I}_n} a_i f_i$ for every $\mathbf{a} = (a_i)_{i\in\mathbb{I}_n} \in \mathbb{C}^n$. With the notation of (1), the frame operator of $\mathcal{F}$ is

$$S_\mathcal{F} = T^*_\mathcal{F} T_\mathcal{F} = \sum_{i\in\mathbb{I}_n} f_i \otimes f_i \in \mathcal{M}_d(\mathbb{C})^+.$$

Notice that, given $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ then $\langle S_\mathcal{F} x, x\rangle = \sum_{i\in\mathbb{I}_n} |\langle x, f_i\rangle|^2$ for every $x \in \mathbb{C}^d$. Hence, $\mathcal{F}$ is a frame if and only if $S_\mathcal{F} \in \mathcal{G}l(d)^+$ and in this case $A_\mathcal{F}\|x\|^2 \le \langle S_\mathcal{F} x, x\rangle \le B_\mathcal{F}\|x\|^2$ for every $x \in \mathbb{C}^d$. Therefore, $A_\mathcal{F} = \lambda_{\min}(S_\mathcal{F}) = \|S_\mathcal{F}^{-1}\|^{-1}$ and $\lambda_{\max}(S_\mathcal{F}) = \|S_\mathcal{F}\| = B_\mathcal{F}$. Moreover, $\mathcal{F}$ is tight if and only if $S_\mathcal{F} = \frac{\tau}{d} I_d$, where $\tau = \operatorname{tr} S_\mathcal{F} = \sum_{i\in\mathbb{I}_n} \|f_i\|^2$.

Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ be a frame for $\mathbb{C}^d$. A family $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ is said to be an (alternate) dual of $\mathcal{F}$ if

$$f = \sum_{i\in\mathbb{I}_n} \langle f, g_i\rangle f_i, \quad \text{for every } f \in \mathbb{C}^d.$$

It is easy to see that $\mathcal{G}$ is a dual of $\mathcal{F}$ iff $T^*_\mathcal{F} T_\mathcal{G} = I_d$. Hence, in that case $\mathcal{G}$ is a frame and $\mathcal{F}$ is a dual of $\mathcal{G}$ and we say that $(\mathcal{F}, \mathcal{G})$ is a dual pair of frames for $\mathbb{C}^d$. We shall consider

$$\mathcal{D}(\mathcal{F}) = \{\mathcal{G} : \mathcal{G} \text{ is a dual frame of } \mathcal{F}\}.$$

The so-called canonical dual of $\mathcal{F}$, denoted $\mathcal{F}^\#$, is given by $\mathcal{F}^\# = \{S_\mathcal{F}^{-1} f_i\}_{i\in\mathbb{I}_n}$. It is straightforward to check that $T_{\mathcal{F}^\#} = T_\mathcal{F} S_\mathcal{F}^{-1}$ and then $T^*_\mathcal{F} T_{\mathcal{F}^\#} = S_\mathcal{F} S_\mathcal{F}^{-1} = I_d$ so $\mathcal{F}^\# \in \mathcal{D}(\mathcal{F})$. The canonical dual is a distinguished dual since it possesses several minimality properties. Nevertheless, notice that whenever $d < n$ (i.e. whenever the frame $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ is a redundant set of linear generators) then $\mathcal{D}(\mathcal{F})$ has infinitely many elements and it turns out to have a very rich structure. This is one of the main advantages of the redundant frame $\mathcal{F}$ over a (not necessarily orthonormal) basis $\mathcal{B} = \{v_i\}_{i\in\mathbb{I}_d}$ in $\mathbb{C}^d$, since the set of duals of the latter has only one element, namely $\mathcal{D}(\mathcal{B}) = \{\mathcal{B}^\#\}$.

In their work [5], Benedetto and Fickus introduced a functional defined (on unit norm frames), the so-called frame potential, given by

$$\operatorname{FP}\big(\{f_i\}_{i\in\mathbb{I}_n}\big) = \sum_{i,j\in\mathbb{I}_n} \big|\langle f_i, f_j\rangle\big|^2.$$

One of their major results shows that tight unit norm frames – which form an important class of frames because of their simple reconstruction formulas – can be characterized as (local) minimizers of this functional among unit norm frames. Since then, there has been interest in (local) minimizers of the frame potential within certain classes of frames, since such minimizers can be considered as natural substitutes of tight frames. Notice that, given a family $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ of vectors in $\mathbb{C}^d$, then $\mathrm{FP}(\mathcal{F}) = \operatorname{tr} S_{\mathcal{F}}^2 = \sum_{i\in\mathbb{I}_d} \lambda_i(S_{\mathcal{F}})^2$. Recently, there has been interest in the structure of minimizers of other potentials such as the so-called mean squared error (MSE) given by $\mathrm{MSE}(\mathcal{F}) = \operatorname{tr}(S_{\mathcal{F}}^{-1})$.

These remarks have motivated the analysis of the structure of minimizers of general convex potentials:

Definition 2.1. Let us denote by

$$\mathrm{Conv}(\mathbb{R}_{\geq 0}) = \big\{h : [0,\infty) \to [0,\infty) : h \text{ is a convex function}\big\}$$

and $\mathrm{Conv}_s(\mathbb{R}_{\geq 0}) = \{h \in \mathrm{Conv}(\mathbb{R}_{\geq 0}) : h \text{ is strictly convex}\}$. Following [16] we consider the (generalized) convex potential P_h associated to $h \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$, given by

$$P_h(\mathcal{F}) = \operatorname{tr} h(S_{\mathcal{F}}) = \sum_{i\in\mathbb{I}_d} h\big(\lambda_i(S_{\mathcal{F}})\big) \quad \text{for } \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n},$$

where the matrix $h(S_{\mathcal{F}})$ is defined by means of the usual functional calculus in $\mathcal{M}_d(\mathbb{C})^+$. $\triangle$

In order to deal with these general convex potential we consider the notions of submajorization and log-majorization in the next section.

2.3. Submajorization and log-majorization

Next we briefly describe majorization and log-majorization, two notions from matrix analysis theory that will be used throughout the paper. For a detailed exposition on these relations see [3]. Majorization between vectors is closely related to design problems in frame theory, mainly due to its connection with Schur–Horn theorem (see for example [4,10,1]).

Given $x, y \in \mathbb{R}^d$ we say that x is submajorized by y, and write $x \prec_w y$, if

$$\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow \quad \text{for every } k \in \mathbb{I}_d.$$

If $x \prec_w y$ and $\operatorname{tr} x = \sum_{i=1}^d x_i = \sum_{i=1}^d y_i = \operatorname{tr} y$, we say that x is majorized by y, and write $x \prec y$.

On the other hand we write $x \leqslant y$ if $x_i \leq y_i$ for every $i \in \mathbb{I}_d$. It is a standard exercise to show that $x \leqslant y \Longrightarrow x^\downarrow \leqslant y^\downarrow \Longrightarrow x \prec_w y$. Our interest in majorization is motivated by the relation of this notion with tracial inequalities for convex functions. Indeed, given $x, y \in \mathbb{R}^d$ and $f : I \to \mathbb{R}$ a convex function defined on an interval $I \subseteq \mathbb{R}$ such that $x, y \in I^d$, then (see for example [3]):

1. If one assumes that $x \prec y$, then $\operatorname{tr} f(x) \stackrel{\text{def}}{=} \sum_{i\in\mathbb{I}_d} f(x_i) \le \sum_{i\in\mathbb{I}_d} f(y_i) = \operatorname{tr} f(y)$.
2. If only $x \prec_w y$, but the map f is also increasing, then still $\operatorname{tr} f(x) \le \operatorname{tr} f(y)$.
3. If $x \prec_w y$ and f is a strictly convex function such that $\operatorname{tr} f(x) = \operatorname{tr} f(y)$ then there exists a permutation σ of $\mathbb{I}_d$ such that $y_i = x_{\sigma(i)}$ for $i \in \mathbb{I}_d$.

Remark 2.2. Majorization between vectors in $\mathbb{R}^d$ is intimately related with the class of doubly stochastic $d\times d$ matrices, denoted by $\mathrm{DS}(d)$. Recall that a $d\times d$ matrix $D \in \mathrm{DS}(d)$ if it has non-negative entries and each row sum and column sum equals 1.

It is well known (see [3]) that given $x, y \in \mathbb{R}^d$ then $x \prec y$ if and only if there exists $D \in \mathrm{DS}(d)$ such that $Dy = x$. As a consequence of this fact we see that if $x_1, y_1 \in \mathbb{R}^r$ and $x_2, y_2 \in \mathbb{R}^s$ are such that $x_i \prec y_i$, $i = 1, 2$, then $x = (x_1, x_2) \prec y = (y_1, y_2)$ in $\mathbb{R}^{r+s}$. Indeed, if D_1 and D_2 are the doubly stochastic matrices corresponding the previous majorization relations then $D = D_1 \oplus D_2 \in \mathrm{DS}(r+s)$ is such that $Dy = x$. $\triangle$

Log-majorization between vectors in $\mathbb{R}^d_{\ge 0}$ is a multiplicative analogue of majorization in $\mathbb{R}^d$. Indeed, given $x, y \in \mathbb{R}^d_{\ge 0}$ we say that x is log-majorized by y, denoted $x \prec_{\log} y$, if

$$\prod_{i=1}^{k} x_i^{\downarrow} \le \prod_{i=1}^{k} y_i^{\downarrow} \quad \text{for every } k \in \mathbb{I}_{d-1} \quad \text{and} \quad \prod_{i=1}^{d} x_i^{\downarrow} = \prod_{i=1}^{d} y_i^{\downarrow}. \tag{4}$$

Our interest in log-majorization is also motivated by the relation of this notion with tracial inequalities for convex functions. It is known (see [3]) that

$$\text{if } x, y \in \mathbb{R}^d_{\ge 0}, \quad x \prec_{\log} y \quad \Longrightarrow \quad x \prec_w y.$$

Hence, if $x, y \in \mathbb{R}^d_{\ge 0}$ are such that $x \prec_{\log} y$ then for every convex and increasing function $f : (0, \infty) \to R$ we get that $\operatorname{tr}(f(x)) \le \operatorname{tr}(f(y))$.

3. Two problems in frame theory

In this section we present a detailed description of the two frame problems together with some further motivations, using the notation and terminology from Section 2.

3.1. Optimal perturbation of the canonical dual frame with restrictions

Consider a fixed frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ for $\mathbb{C}^d$, with $n > d$. Then, the set of dual frames $\mathcal{D}(\mathcal{F})$ has a rich structure. It is known that if $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ then $S_{\mathcal{G}} \ge S_{\mathcal{F}^\#} = S_{\mathcal{F}}^{-1}$, with respect to the operator order. This strong inequality explains several optimality properties of the canonical dual frame $\mathcal{F}$. For example, if $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ and $\mathcal{F}^\# = \{f_i^\#\}_{i\in\mathbb{I}_n}$ we have that

$$\sum_{i\in\mathbb{I}_n} \left\| f_i^\# \right\|^2 = \operatorname{tr}(S_{\mathcal{F}^\#}) \le \operatorname{tr}(S_{\mathcal{G}}) = \sum_{i\in\mathbb{I}_n} \|g_i\|^2,$$

with equality if and only if $\mathcal{G} = \mathcal{F}^\#$.

Nevertheless, in applied situations, it is desired to consider numerically stable encoding–decoding schemes derived from the dual pair $(\mathcal{F}, \mathcal{G})$, for some choice of dual frame $\mathcal{G} \in \mathcal{D}(\mathcal{F})$. A possible way out of this situation is as follows: for $t > \operatorname{tr}(S_{\mathcal{F}^\#})$ consider

$$\mathcal{D}_t(\mathcal{F}) = \left\{ \mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n} \in \mathcal{D}(\mathcal{F}) : \sum_{i \in \mathbb{I}_n} \|g_i\|^2 \geq t \right\}$$

and search for optimal duals within $\mathcal{D}_t(\mathcal{F})$ with respect to some measure of optimality (e.g. minimizers of the condition number). It turns out (see [19]) that there exists a distinguished class $\mathcal{OD}_t(\mathcal{F}) \subset \mathcal{D}_t(\mathcal{F})$ such that for every $\mathcal{G}^o \in \mathcal{OD}_t(\mathcal{F})$ and every $\mathcal{G} \in \mathcal{D}_t(\mathcal{F})$ we have the majorization relation $\lambda(S_{\mathcal{G}^o}) \prec_w \lambda(S_{\mathcal{G}})$. This last fact implies several optimality properties of the class $\mathcal{OD}_t(\mathcal{F})$.

Still, in the search for optimal alternative duals for $\mathcal{F}$, there are some properties of the canonical dual frame $\mathcal{F}^\#$ that we may want to retain. In order to preserve some of the minimal features of the canonical dual frame and yet search for numerically stable alternative duals (which are possibly best suited for practical purposes) we introduce the following class of dual frames: set $m = 2d - n$, let $t > \operatorname{tr}(S_{\mathcal{F}^\#})$ and $\varepsilon > 0$ be such that $t - \operatorname{tr}(S_{\mathcal{F}^\#}) \leq \min\{(d-m), d\} \cdot \varepsilon^2$ and define

$$\mathcal{D}_{(t,\varepsilon)}(\mathcal{F}) = \left\{ \mathcal{G} = \{g_j\}_{j \in \mathbb{I}_n} \in \mathcal{D}(\mathcal{F}) : \sum_{j \in \mathbb{I}_n} \|g_j\|^2 \geq t,\ \|T_{\mathcal{G}} - T_{\mathcal{F}^\#}\| \leq \varepsilon \right\}.$$

Hence, we search for optimal duals for $\mathcal{F}$ within $\mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$. As a criteria for optimality, following [19] we search for $\prec_w$-minimizers of the eigenvalues of the frame operators $S_{\mathcal{G}}$ for $\mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$. Hence, we consider the associated set

$$\mathcal{S}\big(\mathcal{D}_{(t,\varepsilon)}(\mathcal{F})\big) \stackrel{\text{def}}{=} \big\{ S_{\mathcal{G}} :\ \mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F}) \big\} \subseteq \mathcal{M}_d(\mathbb{C})^+.$$

As we shall see, there exist $\prec_w$-minimizers within $\mathcal{S}(\mathcal{D}_{(t,\varepsilon)}(\mathcal{F}))$; moreover, their spectral and geometrical features can be explicitly computed. We point out that the structure of optimal duals depends both on the norm restriction (of the frame elements of $\mathcal{G}$) and on the operator norm distance restriction. As a first step in our analysis, we obtain an explicit representation of the frame operators of the elements of $\mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$.

Proposition 3.1. *Let $\mathcal{F} = \{f_j\}_{j \in \mathbb{I}_n}$ be a frame for $\mathbb{C}^d$ and set $m = 2d - n$. Let $t > t_0 = \operatorname{tr}(S_{\mathcal{F}^\#})$ and $\varepsilon > 0$ be such that $t - t_0 \leq \min\{(d-m), d\} \cdot \varepsilon^2$. Then*

$$\mathcal{S}\big(\mathcal{D}_{(t,\varepsilon)}(\mathcal{F})\big) = \big\{ S_{\mathcal{F}^\#} + B :\ B \in \mathcal{M}_d(\mathbb{C})^+,\ \operatorname{tr}(B) \geq t - t_0,\ \|B\| \leq \varepsilon^2,\ \operatorname{rk}(B) \leq d - m \big\}.$$

Proof. Let $\mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$ and notice that then

$$T_{\mathcal{G}}^* T_{\mathcal{F}} = T_{\mathcal{F}^\#}^* T_{\mathcal{F}} = I_d \quad \Longrightarrow \quad \big(T_{\mathcal{G}}^* - T_{\mathcal{F}^\#}^*\big) T_{\mathcal{F}} = 0.$$

Hence, if we let $A = T_{\mathcal{G}} - T_{\mathcal{F}\#} : \mathbb{C}^d \to \mathbb{C}^n$ then $A^* T_{\mathcal{F}} = 0$ which implies that $T_{\mathcal{F}}^* A = 0$ and hence $R(A) \subseteq \ker(T_{\mathcal{F}}^*) = \ker(T_{\mathcal{F}\#}^*)$. Therefore, $T_{\mathcal{G}} = T_{\mathcal{F}\#} + A$ and

$$T_{\mathcal{G}}^* T_{\mathcal{G}} = T_{\mathcal{F}\#}^* T_{\mathcal{F}\#} + T_{\mathcal{F}\#}^* A + A^* T_{\mathcal{F}\#} + A^* A = T_{\mathcal{F}\#}^* T_{\mathcal{F}\#} + A^* A$$

that is, $S_{\mathcal{G}} = S_{\mathcal{F}\#} + B$, where $B = A^*A \in \mathcal{M}_d(\mathbb{C})^+$. Since $R(A) \subseteq \ker(T_{\mathcal{F}}^*)$ then $\operatorname{rk}(B) = \operatorname{rk}(A) \leq n - d = d - m$ and $\|B\| = \|A^*A\| = \|T_{\mathcal{G}} - T_{\mathcal{F}\#}\|^2 \leq \varepsilon^2$. Notice that $\operatorname{tr}(S_{\mathcal{G}}) = \sum_{j\in\mathbb{I}_n} \|g_j\|^2 \geq t$ which shows that $\operatorname{tr}(B) = \operatorname{tr}(S_{\mathcal{G}}) - \operatorname{tr}(S_{\mathcal{F}\#}) \geq t - \operatorname{tr}(S_{\mathcal{F}\#})$. Incidentally, notice that the existence of such B implies the restriction on t in the statement, since

$$\operatorname{tr}(B) \leq \operatorname{rk}(B) \cdot \|B\| \quad \Longrightarrow \quad t - \operatorname{tr}(S_{\mathcal{F}\#}) \leq \operatorname{tr}(B) \leq \min\big\{(d-m), d\big\} \cdot \varepsilon^2. \tag{5}$$

In order to show the converse inclusion, let $B \in \mathcal{M}_d(\mathbb{C})^+$ be such that $\operatorname{tr}(B) \geq t - \operatorname{tr}(S_{\mathcal{F}\#})$, $\|B\| \leq \varepsilon^2$ and $\operatorname{rk}(B) \leq d - m = n - d = \dim\ker T_{\mathcal{F}}^*$. Then, we can factor $B = A^*A$ where $A : \mathbb{C}^d \to \mathbb{C}^n$ is such that $R(A) \subseteq \ker(T_{\mathcal{F}}^*)$, so that $T_{\mathcal{F}}^* A = 0$ and $A^* T_{\mathcal{F}} = 0$. Let $\mathcal{G} = \{(T_{\mathcal{F}\#} + A)^* e_i\}_{i\in\mathbb{I}_n}$, where $\{e_i\}_{i\in\mathbb{I}_n}$ denotes the canonical basis of $\mathbb{C}^n$. Thus, $T_{\mathcal{G}} = T_{\mathcal{F}\#} + A$ and hence $T_{\mathcal{G}}^* T_{\mathcal{F}} = I_d$, $S_{\mathcal{G}} = S_{\mathcal{F}\#} + B$; in particular, $\sum_{i\in\mathbb{I}_n} \|g_i\|^2 = \operatorname{tr}(S_{\mathcal{G}}) = \operatorname{tr}(S_{\mathcal{F}\#}) + \operatorname{tr}(B) \geq t$ and $\|T_{\mathcal{G}} - T_{\mathcal{F}}^{\#}\| = \|A\| = \|B\|^{1/2} \leq \varepsilon$. Therefore, $\mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$ is such that $S_{\mathcal{G}} = S_{\mathcal{F}\#} + B$. $\square$

Remark 3.2. With the notation of Proposition 3.1, by Eq. (5) and a straightforward construction of a matrix B with the required parameters, we see that the relation

$$t - \operatorname{tr}(S_{\mathcal{F}\#}) \leq \min\big\{(d-m), d\big\} \cdot \varepsilon^2$$

between the parameters t and ε is necessary and sufficient for $\mathcal{D}_{(t,\varepsilon)}(\mathcal{F}) \neq \emptyset$. $\triangle$

3.2. Optimal perturbations by equivalent frames

Fix a frame $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ for $\mathbb{C}^d$, with $n > d$. Hence, $\mathcal{F}$ is a redundant family of linear generators for $\mathbb{C}^d$; in other words, $T_{\mathcal{F}} : \mathbb{C}^d \to \mathbb{C}^n$ is an injective transformation such that $R(T_{\mathcal{F}}) \subset \mathbb{C}^n$ is a proper subspace. These last facts can be used to develop some simple linear tests in order to check whether a sequence $\mathbf{a} = (a_i)_{i\in\mathbb{I}_n} \in \mathbb{C}^n$ is the sequence of frame coefficients $T_{\mathcal{F}}(f) = (\langle f, f_i\rangle)_{i\in\mathbb{I}_n}$ for some (unique) $f \in \mathbb{C}^d$ or whether it has been corrupted (e.g. due to noise in the communication channel). Indeed, given a linear relation $\sum_{i\in\mathbb{I}_n} \alpha_i f_i = 0$ of the family $\mathcal{F}$, we get a linear test $\varphi[(a_i)_{i\in\mathbb{I}_n}] = \sum_{i\in\mathbb{I}_n} \overline{\alpha_i} a_i = 0$ for the sequence $\mathbf{a}$; moreover, we can consider a complete set of tests $\varphi_1, \ldots, \varphi_{n-d} \in (\mathbb{C}^n)^*$ such that, given $\mathbf{a} = (a_i)_{i\in\mathbb{I}_n} \in \mathbb{C}^n$ then $\mathbf{a}$ lies in $R(T_{\mathcal{F}})$ if and only if $\varphi_i(\mathbf{a}) = 0$ for $i \in \mathbb{I}_{n-d}$ (based on the linear relations of the family $\mathcal{F}$).

Hence, the linear relations among the elements of $\mathcal{F}$ play an important role in this context. On the other hand, the numerical stability of the frame $\mathcal{F}$ is also an important feature in practice. Hence, there are situations in which we want to improve the stability of the frame $\mathcal{F}$ (measured in terms of the spread the eigenvalues of its frame operator) while preserving the linear relations among the frame elements. It is well known that if a family $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ has the same linear relations as $\mathcal{F}$ then there exists an invertible linear operator $V \in \mathcal{G}l(d)$ such that $\mathcal{G} = V \cdot \mathcal{F} = \{V f_i\}_{i\in\mathbb{I}_n}$. In this case, following [2] we say that $\mathcal{G} = V \cdot \mathcal{F} = \{V f_i\}_{i\in\mathbb{I}_n}$ and $\mathcal{F}$ are equivalent frames.

As a first step, we can search for an invertible operator $V \in \mathcal{G}l(d)$ such that the frame $V \cdot \mathcal{F}$ is optimal with respect to the spread of the eigenvalues of its frame operator. It turns out that a solution to this (unrestricted) problem is $V = S_{\mathcal{F}}^{-1/2}$ so that $V \cdot \mathcal{F} = \{S_{\mathcal{F}}^{-1/2} f_i\}_{i\in\mathbb{I}_n}$ is the associated Parseval frame (with minimal spread of its eigenvalues). This solution, although optimal, might lie *away* from the original frame $\mathcal{F}$. In case we rather want to consider *perturbations* of $\mathcal{F}$, preserving their linear relations and yet improving its numerical performance, we can search for invertible operators V such that the equivalent frame $V \cdot \mathcal{F}$ is optimal in the previous sense, under some restrictions on V. Hence, given a frame $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ we introduce the following set of controlled perturbations by equivalent frames: given $0 < \delta < 1$ and $s \in [(1-\delta)^d, (1+\delta)^d]$ then consider

$$\mathcal{P}_{(s,\delta)}(\mathcal{F}) \stackrel{\text{def}}{=} \big\{V \cdot \mathcal{F} = \{V f_i\}_{i\in\mathbb{I}_n} :\ V \in \mathcal{G}l(d),\ \|V - I\| \le \delta,\ \big|\det(V)\big| \ge s\big\}. \tag{6}$$

Our main problem is to compute the structure of optimal perturbations $V\cdot\mathcal{F} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$, in the sense that they minimize the spread of the eigenvalues of the frame operators within this class. We point out that in case that we drop the condition on the determinant of V in the definition of $\mathcal{P}_{(s,\delta)}(\mathcal{F})$ above then the optimal perturbations are obtained in terms of homothecies (i.e. for $V = (1-\delta)I$) and thus the problem has no interest in that case (see the comments after Remark 5.3).

In order to compute the structure of optimal perturbations $V \cdot \mathcal{F} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$ we also introduce

$$\mathcal{S}\big(\mathcal{P}_{(s,\delta)}(\mathcal{F})\big) \stackrel{\text{def}}{=} \big\{S_{\mathcal{G}} : G \in \mathcal{P}_{(s,\delta)}(\mathcal{F})\big\}. \tag{7}$$

It is straightforward to check that

$$\mathcal{S}\big(\mathcal{P}_{(s,\delta)}(\mathcal{F})\big) = \big\{V S_{\mathcal{F}} V^* :\ V \in \mathcal{G}l(d),\ \|V - I\| \le \delta,\ \big|\det(V)\big| \ge s\big\}. \tag{8}$$

As we shall see, there exist $\prec_w$ minimizers within $\mathcal{S}(\mathcal{P}_{(s,\delta)}(\mathcal{F}))$; moreover, their spectral and geometrical features can be explicitly computed.

4. Optimal perturbation of the canonical dual frame

In this section, given a frame $\mathcal{F}$, we consider a matrix model for the design of optimal perturbations of the canonical dual frame $\mathcal{F}^{\#}$. In Section 4.1, using tools from matrix analysis, we show that there are optimal solutions within our abstract model. Then, in Section 4.2 we apply these results to the initial frame design problem.

4.1. A matrix model for $\mathcal{S}(\mathcal{D}_{(t,\varepsilon)}(\mathcal{F}))$

In order to deal with the problem of existence of $\prec_w$ minimizers in $\mathcal{S}(\mathcal{D}_{(t,\varepsilon)}(\mathcal{F}))$ for a frame $\mathcal{F}$, we introduce the following set motivated by Proposition 3.1: given $S \in \mathcal{M}_d(\mathbb{C})^+$, $t > t_0 \overset{\text{def}}{=} \operatorname{tr}(S)$, $\varepsilon > 0$ and $m \in \mathbb{Z}$ with $m \le d-1$ such that $t - t_0 \le \min\{(d-m), d\} \cdot \varepsilon$, we define

$$U_{(t,\varepsilon)}(S,m) \overset{\text{def}}{=} \left\{S + B : \ B \in \mathcal{M}_d(\mathbb{C})^+, \ \operatorname{tr}(B) \ge t - t_0, \ \|B\| \le \varepsilon, \ \operatorname{rk}(B) \le d - m\right\}. \tag{9}$$

Hence, Proposition 3.1 states (using the notation in that result) that

$$\mathcal{S}\big(\mathcal{D}_{(t,\varepsilon)}(\mathcal{F})\big) = U_{(t,\varepsilon^2)}(S_{\mathcal{F}^\#}, m).$$

In order to deal with the spectral structure of optimal elements in $U_{(t,\varepsilon)}(S,m)$ the following (rather naive) vector model will be useful: given $\lambda \in (\mathbb{R}^d)^\downarrow$, $t > t_0 \overset{\text{def}}{=} \operatorname{tr}(\lambda)$, $\varepsilon > 0$ and $m \in \mathbb{Z}$ with $m \le d-1$ such that $t - t_0 \le \min\{(d-m), d\} \cdot \varepsilon$ we define

$$\Lambda_{(t,\varepsilon)}(\lambda, m) = \left\{\lambda + \mu : \ 0 \le \mu_i \le \varepsilon, \ \operatorname{tr}\mu \ge t - t_0, \ \operatorname{supp}(\mu) \le d - m\right\} \subseteq \mathbb{R}^d_{\ge 0}$$

where $\operatorname{supp}(\mu) = \|\mu\|_0 = \#\{j \in \mathbb{I}_d : \ \mu_j \ne 0\}$.

Remark 4.1. In [19] (also see [20,21]) we considered a simpler version of the set $\Lambda_{(t,\varepsilon)}(\lambda, m)$ defined above. Indeed, given $\lambda = (\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d)^\downarrow$, $t \ge t_0 = \operatorname{tr}(\lambda)$ we considered the set

$$\Lambda_t(\lambda) = \left\{\lambda + \mu : \ \mu \in \mathbb{R}^d_{\ge 0}, \ \operatorname{tr}(\mu) \ge t - t_0\right\}.$$

In this context (see [19]) we showed that there exists a unique $\nu(\lambda, t) \in \Lambda_t(\lambda)$ such that

$$\nu(\lambda,t) = \nu(\lambda,t)^\downarrow \quad \text{and} \quad \nu(\lambda,t) \prec_w \nu \quad \text{for every } \nu \in \Lambda_t(\lambda),$$

i.e., $\nu(\lambda, t)$ is a $\prec_w$-minimizer in $\Lambda_t(\lambda)$. Moreover, $\nu(\lambda, t)$ can be explicitly computed as follows: let $h : [\lambda_d, \infty) \to \mathbb{R}_{\ge 0}$ be given by $h(x) = \sum_{i\in\mathbb{I}_d} (x - \lambda_i)^+$, where α^+ stands for the positive part of $\alpha \in \mathbb{R}$. Then, h is a continuous and strictly increasing function in its domain, and for $t \ge t_0$ there exists a unique $c_\lambda(t) \in [\lambda_d, \infty)$ such that $h(c_\lambda(t)) = t - t_0$. Then, with this notation we have that $\nu(\lambda, t) = (\lambda_i + (c_\lambda(t) - \lambda_i)^+)_{i\in\mathbb{I}_d} \in \mathbb{R}^d$ i.e.

$$\begin{aligned} &\nu(\lambda,t) = c_\lambda(t) \cdot \mathbb{1}_d \quad \big(\text{if } c_\lambda(t) > \lambda_1\big) \quad \text{or} \\ &\nu(\lambda,t) = \big(\lambda_1, \ldots, \lambda_r, c_\lambda(t) \cdot \mathbb{1}_{d-r}\big) \quad \big(\text{if } c_\lambda(t) \le \lambda_1\big) \end{aligned}$$

for some $r \in \mathbb{I}_d$. Notice that $\nu(\lambda,t) = \nu(\lambda,t)^\downarrow$ and $\operatorname{tr}(\nu(\lambda,t)) = t$ in any case. Also

$$\nu(\lambda,t)-\lambda=\big(\big(c_\lambda(t)-\lambda_i\big)^+\big)_{i\in\mathbb{I}_d}\quad\Longrightarrow\quad \nu(\lambda,t)-\lambda=\big(\nu(\lambda,t)-\lambda\big)^\uparrow. \tag{10}$$

Moreover, $\nu(\lambda,t)$ is determined as the unique $\nu\in\Lambda_t(\lambda)$ such that ν is a $\prec_w$ minimizer in $\Lambda_t(\lambda)$ and such that $\nu-\lambda=(\nu-\lambda)^\uparrow$. $\triangle$

The following lemma is a direct consequence of the results from [19] described in Remark 4.1 above.

Lemma 4.2. *Let $\lambda=(\lambda_i)_{i\in\mathbb{I}_d}\in(\mathbb{R}^d)^\downarrow$, and $t>t_0\stackrel{\text{def}}{=}\operatorname{tr}\lambda$. Assume that $r\in\mathbb{I}_{d-1}$ and $c>0$ are such that the vector $\gamma\stackrel{\text{def}}{=}(\lambda_1,\ldots,\lambda_r,c\mathbb{1}_{d-r})$ satisfies that*

$$\lambda_r\geq c\geq\lambda_{r+1}\quad\big(\textit{so that } \gamma=\gamma^\downarrow\succcurlyeq\lambda\big)\quad \textit{and}\quad \operatorname{tr}\gamma=t. \tag{11}$$

Then, in this case we have that $c=c_\lambda(t)$ and $\gamma=\nu(\lambda,t)$. $\square$

The following statement finds a minimum for submajorization in the set $\Lambda_{(t,\varepsilon)}(\lambda,0)$. Note that $\Lambda_{(t,\varepsilon)}(\lambda,m)=\Lambda_{(t,\varepsilon)}(\lambda,0)$ for every $m\leq 0$.

Theorem 4.3. *Let $\lambda=(\lambda_i)_{i\in\mathbb{I}_d}\in(\mathbb{R}^d)^\downarrow$ and $t_0=\operatorname{tr}\lambda$. Let $\varepsilon>0$ and $t\geq t_0$ be such that $t-t_0\leq d\cdot\varepsilon$. Then there exists $\rho\in\Lambda_{(t,\varepsilon)}(\lambda,0)$, such that*

$$\operatorname{tr}(\rho)=t\quad \textit{and}\quad \rho\prec_w\gamma\quad \textit{for every } \gamma\in\Lambda_{(t,\varepsilon)}(\lambda,0).$$

In this case we can choose a unique $\rho=\rho_{(t,\varepsilon)}(\lambda,0)$ as above and such that $\rho-\lambda=(\rho-\lambda)^\uparrow$. Moreover, this vector ρ also satisfies that $\rho=\rho^\downarrow\in(\mathbb{R}^d)^\downarrow$.

Proof. Assume first that $\lambda\in(\mathbb{R}^d_{\geq 0})^\downarrow$. In what follows we consider the notation from Remark 4.1. We shall construct the vector $\rho_{(t,\varepsilon)}(\lambda,0)=\rho=(\rho_i)_{i\in\mathbb{I}_d}$ recursively as follows:

1. If $c_\lambda(t)-\lambda_d\leq\varepsilon$ then just take $\rho=\nu(\lambda,t)$.
2. If $c_\lambda(t)-\lambda_d>\varepsilon$ then we put $\rho_d=\lambda_d+\varepsilon$ and we consider the following new data:

$$\lambda^{(d-1)}=(\lambda_1,\ldots,\lambda_{d-1})\quad\text{and}\quad t^{(d-1)}=t-\rho_d$$
$$\big(\Longrightarrow\ t^{(d-1)}-\operatorname{tr}\lambda^{(d-1)}\leq(d-1)\varepsilon\big).$$

Then go back to the first step, but applied to the pair $(\lambda^{(d-1)},t^{(d-1)})$. $\triangle$

The hypothesis that $t-t_0\leq d\cdot\varepsilon$ assures that this process stops (at some step $d-m+1\in\mathbb{I}_d$) obtaining the outcome $\rho=(\nu(\lambda^{(m)},t^{(m)}),\lambda_{m+1}+\varepsilon,\ldots,\lambda_d+\varepsilon)\in\mathbb{R}^d_{\geq 0}$. That is,

$$\rho=(\lambda_1,\ldots,\lambda_s,c\mathbb{1}_{m-s},\lambda_{m+1}+\varepsilon,\ldots,\lambda_d+\varepsilon)\quad\text{with}\quad 0\leq c-\lambda_m\leq\varepsilon \tag{12}$$

and $\lambda_{s+1} \leq c < \lambda_s$ in case $(\lambda_1, \dots, \lambda_s, c\mathbb{1}_{m-s}) = \nu(\lambda^{(m)}, t^{(m)})$, or

$$\rho = (c\mathbb{1}_m, \lambda_{m+1} + \varepsilon, \dots, \lambda_d + \varepsilon) \quad \text{with} \quad 0 \leq c - \lambda_m \leq \varepsilon, \tag{13}$$

in case $\nu(\lambda^{(m)}, t^{(m)}) = c\mathbb{1}_m$. It is clear that this $\rho \in \Lambda_{(t,\varepsilon)}(\lambda, 0)$ and that $\operatorname{tr} \rho = t$. Moreover, we claim that $\rho = \rho^{\downarrow}$: notice that we only need to show that $c \geq \lambda_{m+1} + \varepsilon$, where $c = c_{\lambda^{(m)}}(t^{(m)})$.

Indeed, since the algorithm did not stop at the pair $(\lambda^{(m+1)}, t^{(m+1)})$ then $c_{\lambda^{(m+1)}}(t^{(m+1)}) > \lambda_{m+1} + \varepsilon$. By Remark 4.1 and Lemma 4.2, it is easy to see that

$$c_{\lambda^{(m+1)}}\big(t^{(m+1)}\big) = c_{\lambda^{(m)}}\big(t^{(m+1)} - c_{\lambda^{(m+1)}}\big(t^{(m+1)}\big)\big).$$

Moreover, since $c_{\lambda^{(m)}}(x)$ is an increasing function then

$$c_{\lambda^{(m)}}\big(t^{(m+1)} - c_{\lambda^{(m+1)}}\big(t^{(m+1)}\big)\big) \leq c_{\lambda^{(m)}}\big(t^{(m+1)} - (\lambda_{m+1} + \varepsilon)\big) = c_{\lambda^{(m)}}\big(t^{(m)}\big) = c.$$

Hence, $\lambda_{m+1} + \varepsilon \leq c_{\lambda^{(m+1)}}(t^{(m+1)}) \leq c$, which shows that $\rho = \rho^{\downarrow}$.

Fix $\gamma \in \Lambda_{(t,\varepsilon)}(\lambda, 0)$ such that $\gamma = \lambda + \mu$, where $\mu = (\mu_i)_{i \in \mathbb{I}_d}$ is such that $0 \leq \mu_i \leq \varepsilon$ for $i \in \mathbb{I}_d$ and $\operatorname{tr}(\mu) \geq t - \operatorname{tr}(\lambda)$. Then, with the notation of Remark 4.1, the truncation $(\gamma_1, \dots, \gamma_m) \in \Lambda_{t^{(m)}}(\lambda^{(m)})$ because $\lambda_i \leq \gamma_i$ for $i \in \mathbb{I}_m$ and

$$\begin{aligned} \sum_{i=m+1}^{d} \gamma_i - \lambda_i \leq (d-m)\varepsilon = \sum_{i=m+1}^{d} \rho_i - \lambda_i \\ \implies \quad \operatorname{tr}(\gamma_1, \dots, \gamma_m) \geq \operatorname{tr}(\rho_1, \dots, \rho_m) = t^{(m)}. \end{aligned}$$

By construction $(\rho_1, \dots, \rho_m) = \nu(\lambda^{(m)}, t^{(m)}) \prec_w (\gamma_1, \dots, \gamma_m)$. Hence, we get

$$\sum_{i=1}^{k} \rho_i \leq \sum_{i=1}^{k} \gamma_i \quad \text{for every } k \in \mathbb{I}_m. \tag{14}$$

On the other hand, if $m + 1 \leq k \leq d$ then

$$\sum_{i=1}^{k} \rho_i = t - \sum_{i=k+1}^{d} \rho_i = t - \sum_{i=k+1}^{d} (\lambda_i + \varepsilon) \leq \operatorname{tr}(\gamma) - \sum_{i=k+1}^{d} (\lambda_i + \mu_i) = \sum_{i=1}^{k} \gamma_i, \tag{15}$$

since $0 \leq \mu_i \leq \varepsilon$ for $i \in \mathbb{I}_d$. Thus, Eqs. (14) and (15) imply that $\rho \prec_w \gamma$, since $\rho = \rho^{\downarrow}$ (even when the entries of γ are not necessarily arranged in non-increasing order). The fact that

$$\rho - \lambda = \big(\nu\big(\lambda^{(m)}, t^{(m)}\big) - \lambda^{(m)}, \varepsilon \cdot \mathbb{1}_{d-m}\big) = (\rho - \lambda)^{\uparrow}$$

follows from Eq. (10) and that $c - \lambda_m \leq \varepsilon$.

Now, assume that $\lambda_d < 0$. Take some $s > -\lambda_d$, so that the translated vector $\lambda + s \cdot \mathbb{1}_d \in (\mathbb{R}^d_{\geq 0})^\downarrow$. Let $\rho = \rho_{(t+d\cdot s,\varepsilon)}(\lambda + s \cdot \mathbb{1}_d, 0)$ be the $\prec_w$-minimizer in $\Lambda_{(t+d\cdot s,\varepsilon)}(\lambda + s \cdot \mathbb{1}_d, 0)$ constructed in the first part of this proof. Since $\Lambda_{(t+d\cdot s,\varepsilon)}(\lambda + s \cdot \mathbb{1}_d, 0) = \Lambda_{(t,\varepsilon)}(\lambda, 0) + s \cdot \mathbb{1}_d$ we see that $\rho - s \cdot \mathbb{1}_d \in \Lambda_{(t,\varepsilon)}(\lambda, 0)$ is a $\prec_w$-minimizer in $\Lambda_{(t,\varepsilon)}(\lambda, 0)$.

Finally, assume that $\rho' \in \Lambda_{(t,\varepsilon)}(\lambda, 0)$ is a $\prec_w$-minimizer in $\Lambda_{(t,\varepsilon)}(\lambda, 0)$ and such that $\rho' - \lambda = (\rho' - \lambda)^\uparrow$. If $\rho = \rho_{(t,\varepsilon)}(\lambda, 0)$ is as before, then: in case $\rho_d = \lambda_d + \varepsilon$ (respectively $\rho'_d = \lambda_d + \varepsilon$) then it is easy to see that also $\rho'_d = \lambda_d + \varepsilon$ (respectively $\rho_d = \lambda_d + \varepsilon$); this observation allows reduce the problem of uniqueness of ρ to the case in which $\rho_d < \varepsilon$ and $\rho'_d < \varepsilon$, where uniqueness of ρ follows from the comments after Eq. (10) in Remark 4.1. $\square$

Remark 4.4. With the notation of Theorem 4.3, notice that the proof of that result shows an explicit and simple algorithm that computes the optimal vector ρ in terms of the vector $\nu(\tilde{\lambda}, \tilde{t})$ described in Remark 4.1, for an explicit (and computable) $\tilde{\lambda} \in \mathbb{R}^r$ and $\tilde{t} \in \mathbb{R}$. Since $\nu(\tilde{\lambda}, \tilde{t})$ can be also computed in terms of a simple algorithm (as described in Remark 4.1), we see that the optimal vector ρ can be effectively computed (with a fast algorithm). $\triangle$

The following result complements Theorem 4.3.

Theorem 4.5. *Let $\lambda \in (\mathbb{R}^d)^\downarrow$ and $t_0 = \operatorname{tr} \lambda$. Let $\varepsilon > 0$, $m \in \mathbb{Z}$ with $m \leq d - 1$ and let $t \geq t_0$ be such that $t - t_0 \leq \min\{(d - m), d\} \cdot \varepsilon$. Then there exists $\rho \in \Lambda_{(t,\varepsilon)}(\lambda, m)$, such that*

$$\operatorname{tr}(\rho) = t \quad \textit{and} \quad \rho \prec_w \gamma \quad \textit{for every } \gamma \in \Lambda_{(t,\varepsilon)}(\lambda, m).$$

Moreover, we can choose a unique $\rho = \rho_{(t,\varepsilon)}(\lambda, m)$ as above such that $\rho - \lambda = (\rho - \lambda)^\uparrow$ (although the entries of ρ may not be arranged in non-increasing order).

Proof. If we assume that $m \leq 0$ then we let $\rho_{(t,\varepsilon)}(\lambda, m) \stackrel{\text{def}}{=} \rho_{(t,\varepsilon)}(\lambda, 0)$. Since in this case $\Lambda_{(t,\varepsilon)}(\lambda, m) = \Lambda_{(t,\varepsilon)}(\lambda, 0)$, Theorem 4.3 implies that $\rho_{(t,\varepsilon)}(\lambda, m)$ has the desired properties.

Assume now that $m \in \mathbb{I}_{d-1}$. Set $\tilde{\lambda} = (\lambda_{m+1}, \ldots, \lambda_d) \in (\mathbb{R}^{d-m})^\downarrow$, $s = t - \sum_{i-1}^m \lambda_i$, and consider $\rho_{(s,\varepsilon)}(\tilde{\lambda}, 0) \in \Lambda_{(s,\varepsilon)}(\tilde{\lambda}, 0) \subseteq \mathbb{R}^{d-m}$ as in Theorem 4.3. Then we define

$$\rho = \rho_{(t,\varepsilon)}(\lambda, m) \stackrel{\text{def}}{=} \big(\lambda_1, \ldots, \lambda_m, \rho_{(s,\varepsilon)}(\tilde{\lambda}, 0)\big) \in \mathbb{R}^d. \tag{16}$$

Clearly, $\rho \in \Lambda_{(t,\varepsilon)}(\lambda, m)$ and $\operatorname{tr}(\rho) = t$. Let $\gamma = \lambda + \mu \in \Lambda_{(t,\varepsilon)}(\lambda, m)$, so that $0 \leq \mu_i \leq \varepsilon$ for $i \in \mathbb{I}_d$, $\operatorname{supp}(\mu) \leq d - m$ and $\operatorname{tr}(\mu) \geq t - t_0$. By Lidskii's (additive) inequality we get that $\lambda + \mu^\uparrow \prec \gamma$.

But $\mu^\uparrow = (0 \cdot \mathbb{1}_m, \tilde{\mu})$, where $\tilde{\mu} = (\tilde{\mu}_i)_{i \in \mathbb{I}_{d-m}} \in (\mathbb{R}^{d-m})^\uparrow$ is such that $0 \leq \tilde{\mu}_i \leq \varepsilon$ for $i \in \mathbb{I}_{d-m}$ and $\operatorname{tr}(\tilde{\mu}) = \operatorname{tr}(\mu) \geq t - t_0 = s - \operatorname{tr}(\tilde{\lambda})$. Hence, $\tilde{\lambda} + \tilde{\mu} \in \Lambda_{(s,\varepsilon)}(\tilde{\lambda}, 0)$ and therefore

$\rho_{(s,\varepsilon)}(\tilde{\lambda}, 0) \prec_w \tilde{\lambda}+\tilde{\mu}$, by Theorem 4.3. Thus, using the fact that we have submajorization by blocks as in Remark 2.2,

$$\rho_{(t,\varepsilon)}(\lambda, m) = \big(\lambda_1, \ldots, \lambda_m, \rho_{(s,\varepsilon)}(\tilde{\lambda}, 0)\big) \prec_w (\lambda_1, \ldots, \lambda_m, \tilde{\lambda} + \tilde{\mu}) = \lambda + \mu^\uparrow \prec \gamma.$$

Finally, notice that the vector ρ as defined in Eq. (16) may be not arranged in non-increasing order. Nevertheless, according to Theorem 4.3

$$\rho_{(t,\varepsilon)}(\lambda, m) - \lambda = \big(0 \cdot \mathbb{1}_m, \rho_{(s,\varepsilon)}(\tilde{\lambda}, 0) - \tilde{\lambda}\big) \in \big(\mathbb{R}^d\big)^\uparrow.$$

Let $\rho' \in \Lambda_{(t,\varepsilon)}(\lambda, m)$ be a $\prec_w$-minimizer in $\Lambda_{(t,\varepsilon)}(\lambda, m)$ and such that $\rho' - \lambda = (\rho' - \lambda)^\uparrow$. Then it is easy to see that $\rho' = (\lambda_1, \ldots, \lambda_m, \rho'')$, where $\rho'' \in \Lambda_{(s,\varepsilon)}(\tilde{\lambda}, 0)$ is a $\prec_w$ minimizer in $\Lambda_{(s,\varepsilon)}(\tilde{\lambda}, 0)$ and such that $\rho'' - \tilde{\lambda} = (\rho'' - \tilde{\lambda})^\uparrow$. Hence, by Theorem 4.3, we conclude that $\rho'' = \rho_{(s,\varepsilon)}(\tilde{\lambda}, 0)$ and therefore $\rho' = \rho_{(t,\varepsilon)}(\lambda, m)$. $\square$

Based on Lidskii's inequality, Theorem 4.5 allows to compute the structure of optimal elements in $U_{(t,\varepsilon)}(S, m)$ from its rather naive model in $\mathbb{R}^d$.

Theorem 4.6. *Let $S \in \mathcal{M}_d(\mathbb{C})^+$, $t > t_0 \overset{\text{def}}{=} \operatorname{tr}(S)$, $\varepsilon > 0$ and $m \in \mathbb{Z}$ with $m \leq d - 1$ such that $t - t_0 \leq \min\{(d - m), d\} \cdot \varepsilon$. Let $\lambda = \lambda(S)$ and $\rho = \rho_{(t,\varepsilon)}(\lambda, m) \in \Lambda_{(t,\varepsilon)}(\lambda, m)$ be the $\prec_w$ minimizer from Theorem 4.5. Then,*

1. *There exists $S_0 \in U_{(t,\varepsilon)}(S, m)$ such that $\lambda(S_0) = \rho^\downarrow$.*
2. *$S_1 \in U_{(t,\varepsilon)}(S, m)$ is such that $\lambda(S_1) \prec_w \lambda(S')$ for every $S' \in U_{(t,\varepsilon)}(S, m) \iff \lambda(S_1) = \rho^\downarrow$.*
3. *If $S + B \in U_{(t,\varepsilon)}(S, m)$ is such that $\lambda(S + B) = \rho^\downarrow$ then there exists an o.n.b. $\{v_j\}_{j\in\mathbb{I}_d}$ for $\mathbb{C}^d$ such that, with the notation of (1),*

$$S = \sum_{j\in\mathbb{I}_d} \lambda_j(S) v_j \otimes v_j \quad \textit{and} \quad B = \sum_{j\in\mathbb{I}_d} \lambda_{d-j+1}(B) v_j \otimes v_j.$$

Proof. Let $\{w_j\}_{j\in\mathbb{I}_d}$ be an o.n.b. for $\mathbb{C}^d$ such that $S = \sum_{j\in\mathbb{I}_d} \lambda_j w_j \otimes w_j$, where $\lambda(S) = \lambda = (\lambda_i)_{i\in\mathbb{I}_d}$. Let $\rho = \rho_{(t,\varepsilon)}(\lambda, m)$ be as in Theorem 4.5 and let $\mu = \rho - \lambda$, so that $\mu = \mu^\uparrow$. Let $B_0 = \sum_{j\in\mathbb{I}_d} \mu_j w_j \otimes w_j$ and notice that then $B_0 \in \mathcal{M}_d(\mathbb{C})^+$, $\|B_0\| \leq \varepsilon$, $\operatorname{tr}(B_0) = t - \operatorname{tr}(S)$ and $\operatorname{rk}(B_0) = \operatorname{supp}(\mu) \leq m - d$, so that $S + B_0 \in U_{(t,\varepsilon)}(S, m)$. Moreover, by construction $\lambda(S + B_0) = \rho^\downarrow$.

Let $S' \in U_{(t,\varepsilon)}(S, m)$ and let $B \in \mathcal{M}_d(\mathbb{C})^+$, $\|B\| \leq \varepsilon$, $\operatorname{rk}(B) \leq d - m$ and $\operatorname{tr}(B) \geq t - \operatorname{tr}(S)$ so that $S' = S + B$. By Lidskii's additive inequality we conclude that $\lambda + \lambda(B)^\uparrow \prec \lambda(S')$. Now, by the hypothesis on B we conclude that $\lambda + \lambda(B)^\uparrow \in \Lambda_{(t,\varepsilon)}(\lambda, m)$. Hence

$$\rho \prec_w \lambda(S) + \lambda(B)^\uparrow \prec \lambda\big(S'\big).$$

Hence, if $S_1 \in U_{(t,\varepsilon)}(S,m)$ is such that $\lambda(S_1) = \rho^\downarrow$ then $\lambda(S_1) \prec_w \lambda(S')$ for every $S' \in U_{(t,\varepsilon)}(S,m)$. Conversely, if $S_1 \in U_{(t,\varepsilon)}(S,m)$ is such that $\lambda(S_1) \prec_w \lambda(S')$ for every $S' \in U_{(t,\varepsilon)}(S,m)$ then $\lambda(S_1) \prec_w \lambda(S_0) \prec_w \lambda(S_1)$ implies that $\lambda(S_1) = \lambda(S_0) = \rho^\downarrow$.

Finally, if $S + B \in U_{(t,\varepsilon)}(S,m)$ is such that $\lambda(S+B) = \rho^\downarrow$ then by Lidskii's additive inequality, the fact that $\lambda(S) + \lambda(B)^\uparrow \in \Lambda_{(t,\varepsilon)}(\lambda,m)$ and Theorem 4.5 we see that

$$\lambda + \lambda(B)^\uparrow \prec_w \lambda(S+B) = \rho^\downarrow \prec_w \lambda + \lambda(B)^\uparrow \quad \Longrightarrow \quad \lambda(S+B) = \big(\lambda + \lambda(B)^\uparrow\big)^\downarrow.$$

Hence, the existence of the o.n.b. $\{v_j\}_{j\in\mathbb{I}_d}$ of item 3 is a consequence of [20, Theorem 8.8] (the case of equality in Lidskii's additive inequality). □

Remark 4.7. The proof of Theorem 4.5 together with Remark 4.4 show that the vector $\rho = \rho_{(t,\varepsilon)}(\lambda,m) \in \Lambda_{(t,\varepsilon)}(\lambda,m)$ as in the statement of Theorem 4.6 can be explicitly computed in terms of a simple (and fast) algorithm depending on t, ε, λ and m. △

4.2. Computation of optimal perturbations of the canonical dual frame

Notation 4.8. Let $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ be a frame for $\mathbb{C}^d$ and set $m = 2d - n$. Let $\varepsilon > 0$ and $t > \operatorname{tr}(S_{\mathcal{F}\#})$ be such that $t - \operatorname{tr}(S_{\mathcal{F}\#}) \le \min\{(d-m), d\} \cdot \varepsilon^2$. Recall that Proposition 3.1 and Eq. (9) imply that

$$\mathcal{S}\big(\mathcal{D}_{(t,\varepsilon)}(\mathcal{F})\big) = U_{(t,\varepsilon^2)}(S_{\mathcal{F}\#}, m). \tag{17}$$

Theorem 4.9. *Consider Notation 4.8. Let $\lambda = \lambda(S_{\mathcal{F}\#})$ and $\rho = \rho_{(t,\varepsilon^2)}(\lambda,m)$ be the $\prec_w$ minimizer for $\Lambda_{(t,\varepsilon^2)}(\lambda,m)$ given in Theorem 4.5. Then,*

1. *There exists $\mathcal{G}_0 \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$ such that $\lambda(S_{\mathcal{G}_0}) = \rho^\downarrow$.*
2. *Fix any $\mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$. Then this $\mathcal{G}$ satisfies that*

$$S_{\mathcal{G}} \prec_w S_{\mathcal{G}'} \quad \text{for every } \mathcal{G}' \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F}) \quad \Longleftrightarrow \quad \lambda(S_{\mathcal{G}}) = \rho^\downarrow.$$

3. *Given $\mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$, then $\lambda(S_{\mathcal{G}}) = \rho^\downarrow \iff \mathcal{G} = \{f_i^\# + k_i\}_{i\in\mathbb{I}_n}$ for some $\mathcal{K} = \{k_i\}_{i\in\mathbb{I}_n}$ such that $T_{\mathcal{F}}^* T_{\mathcal{K}} = 0$ and such there exists an o.n.b. $\{v_j\}_{j\in\mathbb{I}_d}$ for $\mathbb{C}^d$ with*

$$S_{\mathcal{F}\#} = \sum_{j\in\mathbb{I}_d} \lambda_j v_j \otimes v_j \quad \text{and} \quad S_{\mathcal{K}} = \sum_{j\in\mathbb{I}_d} (\rho - \lambda)_j v_j \otimes v_j. \tag{18}$$

In this case $S_{\mathcal{G}} = S_{\mathcal{F}\#} + S_{\mathcal{K}}$; in particular, $S_{\mathcal{F}\#}$ and $S_{\mathcal{G}}$ commute.

Proof. It is a direct consequence of Eq. (17), Theorem 4.6 and the characterization of dual frames given in Proposition 3.1. □

Notice that Theorem 4.9 contains a procedure to compute optimal duals $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$. In what follows, given $h \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ we consider its associated convex potential P_h on finite sequences in $\mathbb{C}^d$ given by $P_h(\mathcal{F}) = \mathrm{tr}(h(S_{\mathcal{F}})) = \sum_{i=1}^d h(\lambda_i(S_{\mathcal{F}}))$.

Corollary 4.10. *Fix an increasing function $h \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$. With the notation and terminology of Theorem 4.9, the following inequality holds:*

$$\sum_{i\in\mathbb{I}_d} h(\rho_i) \leq P_h(\mathcal{G}) \quad \textit{for every } \mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F}), \tag{19}$$

and this lower bound is attained. If we assume further that $h \in \mathrm{Conv}_s(\mathbb{R}_{\geq 0})$, then $\mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$ attains the lower bound in (19) $\iff \mathcal{G} = \{f_i^{\#} + k_i\}_{i\in\mathbb{I}_n}$ for some $\mathcal{K} = \{k_i\}_{i\in\mathbb{I}_n}$ such that $T_{\mathcal{F}}^* T_{\mathcal{K}} = 0$ and such there exists an o.n.b. $\{v_j\}_{j\in\mathbb{I}_d}$ for which Eq. (18) holds.

Proof. It follows from Theorems 4.9 and the standard results of Section 2.3. $\square$

Remark 4.11. There are some $h \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ for which their associated convex potential P_h can be computed in a rather direct way (i.e., without necessarily computing the eigenvalues of the frame operator of the sequence of vector). For example, if $h(x) = x^2$ then $P_h = \mathrm{FP}$ is the so-called frame potential. In this case, given a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ then it is well known that

$$P_h(\mathcal{G}) = \mathrm{FP}(\mathcal{G}) = \sum_{i,j\in\mathbb{I}_n} \left|\langle g_i, g_j\rangle\right|^2.$$

Now, consider a fixed frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ and assume that $\lambda(S_{\mathcal{F}})$ is a known data. Set $m = 2d - n$, let $\varepsilon > 0$ and $t > \mathrm{tr}(S_{\mathcal{F}\#})$ be such that $t - \mathrm{tr}(S_{\mathcal{F}\#}) \leq \min\{(d-m), d\} \cdot \varepsilon^2$. In this case $\lambda(S_{\mathcal{F}\#})$ is also a known data and therefore $\rho \in \mathbb{R}^d$ as in Corollary 4.10 can be explicitly computed (see Remark 4.7). Thus, according to Corollary 4.10 above we get

$$\sum_{i\in\mathbb{I}_d} \rho_i^2 \leq \sum_{i,j\in\mathbb{I}_n} \left|\langle g_i, g_j\rangle\right|^2, \quad \text{for every } \mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F}). \tag{20}$$

The previous inequality provides a quantitative criteria for checking the optimality of $\mathcal{G}$. That is, the closer $\mathrm{FP}(\mathcal{G})$ is to this explicit lower bound, the more concentrated $\lambda(S_{\mathcal{G}})$ is (which is the type of analysis that originally motivated the introduction of the frame potential). Indeed, since $h(x) = x^2$ is strictly convex, Corollary 4.10 and Theorem 4.9 imply that if $\mathcal{G} \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$ attains the lower bound in Eq. (20) then $\lambda(S_{\mathcal{G}})$ has minimal spread (in the sense that is a $\prec_w$-minimizer) among $\lambda(S_{\mathcal{G}'})$ for $\mathcal{G}' \in \mathcal{D}_{(t,\varepsilon)}(\mathcal{F})$. Moreover, in this case the geometrical structure of $S_{\mathcal{G}}$ can be described explicitly in terms of the geometrical structure of $S_{\mathcal{F}}$. $\triangle$

5. Optimal perturbations by equivalent frames

In this section, given a frame $\mathcal{F}$, we consider a matrix model for the design of perturbations of the identity V such that $V \cdot \mathcal{F}$ has the desired optimality properties. In Section 5.1, using tools from matrix analysis, we show that there are optimal solutions within our abstract model. Along the way, we will prove some results that are interesting in their own right, and develop some aspects of the multiplicative Lidskii's inequality with respect to log-majorization (see Appendix A). Then, in Section 5.2 we apply these results to the initial frame design problem.

5.1. A matrix model for $\mathcal{S}(\mathcal{P}_{(s,\delta)}(\mathcal{F}))$

In order tackle the problem of the existence and computation of optimal perturbations by equivalent frames of a fixed frame we introduce the following matrix model: given $S \in \mathcal{M}_d(\mathbb{C})^+$, $0 < \delta < 1$ and $s \in [(1-\delta)^d, (1+\delta)^d]$ then consider

$$\mathcal{O}_{(s,\delta)}(S) = \big\{ VSV^* : \ V \in \mathcal{G}l(d), \ \|V - I\| \le \delta, \ \big|\det(V)\big| \ge s \big\}. \tag{21}$$

With the notation of Section 3.2 and Eqs. (6), (7), since $S_{V\cdot\mathcal{F}} = VS_{\mathcal{F}}V^*$ then Eq. (8) becomes $\mathcal{S}(\mathcal{P}_{(s,\delta)}(\mathcal{F})) = \mathcal{O}_{(s,\delta)}(S_{\mathcal{F}})$.

The following result, that is the multiplicative analogue of Lidskii's (additive) inequality, will play a key role in our work on frames. We develop its proof in Appendix A.

Theorem 5.1. *Let $S \in \mathcal{G}l(d)^+$ and let $\gamma \in (\mathbb{R}^d_{>0})^\downarrow$. Then, for every $V \in \mathcal{G}l(d)$ such that $\lambda(V^*V) = \gamma$ we have that*

$$\lambda(S) \circ \gamma^\uparrow \prec_{\log} \lambda\big(VSV^*\big) \prec_{\log} \lambda(S) \circ \gamma \in \big(\mathbb{R}^d_{>0}\big)^\downarrow. \tag{22}$$

Moreover, if $\lambda(VSV^*) = (\lambda(S) \circ \gamma^\uparrow)^\downarrow$ (resp. $\lambda(VSV^*) = \lambda(S) \circ \gamma$) then S and $|V|$ commute and moreover there exists an o.n.b. $\{v_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that

$$S = \sum_{i\in\mathbb{I}_d} \lambda_i(S) v_i \otimes v_i \quad \text{and} \quad |V| = \sum_{i\in\mathbb{I}_d} \gamma_{d+1-i}^{1/2} v_i \otimes v_i \tag{23}$$

(resp. $S = \sum_{i\in\mathbb{I}_d} \lambda_i(S) v_i \otimes v_i$ and $|V| = \sum_{i\in\mathbb{I}_d} \gamma_i^{1/2} v_i \otimes v_i$). $\square$

The previous result allows to show the existence of optimal (minimal spectral spread) elements in $\mathcal{O}_{(s,\delta)}(S)$ (see Eq. (21)); we further compute their geometric structure.

Theorem 5.2. *Let $S \in \mathcal{G}l(d)^+$, $0 < \delta < 1$ and let $s \in [(1-\delta)^d, (1+\delta)^d]$. Define the following data: $\lambda = \log \lambda(S) = (\log \lambda_i(S))_{i\in\mathbb{I}_d} \in (\mathbb{R}^d)^\downarrow$,*

$$t = \log\bigg(\frac{s^2 \cdot \det(S)}{(1-\delta)^{2d}}\bigg) \ge \operatorname{tr}(\lambda) \quad \text{and} \quad \varepsilon = 2\log\bigg(\frac{1+\delta}{1-\delta}\bigg) > 0.$$

Let $\rho = \rho_{(t,\varepsilon)}(\lambda, 0) \in \Lambda_{(t,\varepsilon)}(\lambda, 0)$ be as in Theorem 4.3. Denote by $\mu = \mu_{(s,\delta)}(S)$ the vector

$$\mu = (1-\delta)^2 \cdot \exp \rho = \left((1-\delta)^2 e^{\rho_i}\right)_{i\in\mathbb{I}_d} \in \left(\mathbb{R}^d_{>0}\right)^{\downarrow}. \tag{24}$$

Then,

1. *There exists $S_0 \in \mathcal{O}_{(s,\delta)}(S)$ such that $\lambda(S_0) = \mu$.*
2. *For every $\tilde{S} \in \mathcal{O}_{(s,\delta)}(S)$ we have that*

$$\prod_{i=1}^{k} \mu_i \le \prod_{i=1}^{k} \lambda_i(\tilde{S}) \quad \textit{for every } k \in \mathbb{I}_d. \tag{25}$$

3. *Given $\tilde{S} = VSV^* \in \mathcal{O}_{(s,\delta)}(S)$ then equality holds in Eq. (25) for every $k \in \mathbb{I}_d$ (i.e. $\mu = \lambda(\tilde{S})$) $\iff |\det(V)| = s$ and there exists an o.n.b. $\{v_i\}_{i\in\mathbb{I}_d}$ for $\mathbb{C}^d$ such that*

$$S = \sum_{i\in\mathbb{I}_d} \lambda_i(S) v_i \otimes v_i \quad \textit{and} \quad V^*V = \sum_{i\in\mathbb{I}_d} \frac{\mu_i}{\lambda_i(S)} v_i \otimes v_i. \tag{26}$$

Proof. First notice that

$$t - \operatorname{tr}(\lambda) = 2\log\left(\frac{s}{(1-\delta)^d}\right) \le d \cdot 2\log\left(\frac{1+\delta}{1-\delta}\right) = d \cdot \varepsilon.$$

Hence, we can compute $\rho_{(t,\varepsilon)}(\lambda, 0)$ as in Theorem 4.3.

Let $\{v_i\}_{i\in\mathbb{I}_d}$ be an o.n.b. for $\mathbb{C}^d$ such that $Sv_i = \lambda_i(S)v_i$ for $i \in \mathbb{I}_d$. Then we define

$$V_0 = \sum_{i\in\mathbb{I}_d} \left(\frac{\mu_i}{\lambda_i(S)}\right)^{1/2} v_i \otimes v_i \in \mathcal{G}l(d)^+ \quad \text{and} \quad S_0 = V_0 S V_0. \tag{27}$$

It is clear that $\lambda(S_0) = \mu$. Since $\rho - \lambda = (\rho - \lambda)^{\uparrow}$ by Theorem 4.3, then

$$\log \lambda^{\uparrow}\left(V_0^2\right) = 2\log(1-\delta) \cdot \mathbb{1}_d + \rho - \lambda. \tag{28}$$

Hence, for any $i \in \mathbb{I}_d$

$$\begin{aligned} & 2\log(1-\delta) \le \log \lambda_i\left(V_0^2\right) \le 2\log(1-\delta) + \varepsilon = 2\log(1+\delta) \\ \implies \quad & (1-\delta)^2 \le \lambda_i\left(V_0^2\right) \le (1+\delta)^2, \end{aligned}$$

which is equivalent to $\|V_0 - I\| \le \delta$. Also, $\det(V_0^2) = (1-\delta)^{2d} \det(S)^{-1} \exp(\operatorname{tr}(\rho)) = s^2$ so that $S_0 \in \mathcal{O}_{(s,\delta)}(S)$ with $\lambda(S_0) = \mu$.

In order to show items 2 and 3, let us first assume that $V \in \mathcal{G}l(d)^+$. Then, by Theorem 5.1,

$$\lambda(S) \circ \lambda\big(V^2\big)^{\uparrow} \prec_{\log} \lambda(VSV) \quad \text{and} \quad \big(\lambda(S) \circ \lambda\big(V^2\big)^{\uparrow}\big)^{\downarrow} = \lambda(VSV)$$

if and only if there exists an o.n.b. such that Eq. (23) holds. Therefore

$$\log\big(\lambda(S)\big) + \log\big(\lambda\big(V^2\big)\big)^{\uparrow} = \log\big(\lambda(S) \circ \lambda\big(V^2\big)^{\uparrow}\big) \prec \log\big(\lambda(VSV)\big). \tag{29}$$

Assume further that $\det(V) \geq s$ and $\|I - V\| \leq \delta$. Then $(1-\delta)^2 \leq \lambda_i(V^2) \leq (1+\delta)^2$ and hence $2\log(1-\delta) \leq \log(\lambda_i(V^2)) \leq 2\log(1+\delta)$ for every $i \in \mathbb{I}_d$. On the other hand

$$\operatorname{tr} \log\big(\lambda\big(V^2\big)\big) = \log\big(\det V^2\big) \geq 2\log(s).$$

Hence $\gamma = \log(\lambda(V^2))^{\uparrow} - 2\log(1-\delta) \cdot \mathbb{1} \in (\mathbb{R}^d_{\geq 0})^{\uparrow}$ is such that

$$\begin{aligned} &\operatorname{tr}(\gamma) \geq 2\log\left(\frac{s}{(1-\delta)^d}\right) \quad \text{and} \\ &0 \leq \gamma_i \leq 2\big(\log(1+\delta) - \log(1-\delta)\big) = 2\log\left(\frac{1+\delta}{1-\delta}\right), \quad i \in \mathbb{I}_d. \end{aligned}$$

Thus, using the notation in the statement we see that

$$\log\big(\lambda(S)\big) + \gamma = \lambda + \gamma \in \Lambda_{(t,\varepsilon)}(\lambda, 0) \quad \Longrightarrow \quad \rho \prec_w \lambda + \gamma.$$

Therefore, in this case we have that $\rho + 2\log(1-\delta) \cdot \mathbb{1} \prec_w \lambda + \log(\lambda(V^2))^{\uparrow}$. These facts together with Eq. (29) imply that

$$\rho + 2\log(1-\delta) \cdot \mathbb{1} \prec_w \log\big(\lambda(S)\big) + \log\big(\lambda\big(V^2\big)\big)^{\uparrow} \prec \log\big(\lambda(VSV)\big).$$

Since the exponential function is increasing, the submajorization relation above implies Eq. (25). Moreover, if equality holds in Eq. (25) for every $k \in \mathbb{I}_d$ then, by the previous majorization relations, we conclude that $(1-\delta)^2 \exp(\rho) = (\lambda(S) \circ \lambda(V^2)^{\uparrow})^{\downarrow} = \lambda(VSV)$. Hence, by Theorem 5.1, we see that there exists an o.n.b. $\{v_i\}_{i\in\mathbb{I}_d}$ that satisfies Eq. (26). Conversely, notice that if there exists an o.n.b. $\{v_i\}_{i\in\mathbb{I}_d}$ for which Eq. (26) holds then it is straightforward to show that equality holds in Eq. (25) for $k \in \mathbb{I}_d$.

Finally, if $V \in \mathcal{G}l(d)$ is arbitrary, the result follows from the positive case by taking the polar decomposition $V = U|V|$, using that $\lambda(VSV^*) = \lambda(|V|S|V|)$ and that $\|I - V\| \geq \|I - |V|\|$ by Lidskii's theorem for singular values. Then $|V|S|V| \in \mathcal{O}_{(s,\delta)}(S)$ and hence $\mu \prec_{\log} \lambda(VSV^*) = \lambda(|V|S|V|)$. $\square$

Remark 5.3. Consider the notation of Theorem 5.2 and let V_0 and S_0 be as in Eq. (27).

1. Then, $S_0 \in \mathcal{O}_{(s,\delta)}(S)$ is such that $\lambda(S_0) = \mu$.
2. If $V \in \mathcal{G}l(d)$ is such that $VSV^* \in \mathcal{O}_{(s,\delta)}(S)$ and $\lambda(VSV^*) = \mu$ then by Eq. (26) there exist unitaries U, W such that $WS = SW$ and $V = UV_0W$, so that $\|UV_0W - I\| \leq \delta$. It could be the case that $W \neq I$ and $U \neq W^*$.

3. Given a unitary U then, even when $\lambda((UV_0)S(UV_0)^*) = \mu$, it could be the case that $UV_0 \cdot \mathcal{F} \notin \mathcal{O}_{(s,\delta)}(S)$ (e.g. when $\|U - I\|$ is not sufficiently small).
4. From the proof of Theorem 5.2 it is possible to derive conditions that guarantee that the distance between V_0 and the identity will equal δ. For example, if we assume that

$$\frac{\lambda_1(S)}{\lambda_d(S)} \geq \left(\frac{1+\delta}{1-\delta}\right)^2, \tag{30}$$

then $\|V_0 - I\| = \delta$. Indeed, following the notation of Theorem 5.2 and its proof, the condition in Eq. (30) is equivalent to

$$\lambda_1 - \lambda_d = \log \lambda_1(S) - \log \lambda_d(S) \geq \varepsilon. \tag{31}$$

Moreover, notice that $\|V_0 - I\| = \delta$ if and only if $\lambda_1(V_0) = 1 + \delta$ or $\lambda_d(V_0) = 1 - \delta$. By Eq. (28) this is possible if and only if $\rho_1 = \lambda_1$ or if $\rho_d = \lambda_d + \varepsilon$. If we assume that $\rho_d < \lambda_d + \varepsilon$, then $\rho = \nu(\lambda, t)$ by construction (see Theorem 4.3). Then, $\rho_1 = \nu(\lambda, t)_1 = \lambda_1$; otherwise, we must have

$$\lambda_1 \leq \rho_1 = \nu(\lambda, t)_1 = \nu(\lambda, t)_d = \rho_d < \lambda_d + \varepsilon,$$

which contradicts Eq. (31). $\triangle$

Consider the notation of Theorem 5.2 and set $s = (1 - \delta)^d$. In this case we have that

$$\mathcal{O}_{(s,\delta)}(S) = \{VSV^* : \ V \in \mathcal{G}l(d), \ \|V - I\| \leq \delta\}. \tag{32}$$

By inspection of the proof of Theorem 5.2 we get that $V_0 = (1 - \delta)I$, $S_0 = (1 - \delta)^2 S \in \mathcal{O}_{(s,\delta)}(S)$ and

$$(1 - \delta)^2 S \leq VSV^* \quad \text{for } VSV^* \in \mathcal{O}_{(s,\delta)}(S). \tag{33}$$

The operator inequality in (33) implies that $(1-\delta)^2 S$ has minimal spectral spread (with respect to log-majorization) within $\mathcal{O}_{(s,\delta)}(S)$ and the problem has no interest in this case.

5.2. Computation of optimal perturbations by equivalent frames

We begin with the following result, which is a consequence of Theorem 5.1, and the relations between submajorization and increasing convex functions described in Section 2.3.

Theorem 5.4. *Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be a frame for $\mathbb{C}^d$, with frame operator $S_\mathcal{F} \in \mathcal{G}l(d)^+$, and fix an increasing function $h \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$. Given $\gamma \in (\mathbb{R}^d_{>0})^\downarrow$ then,*

1. *If $V \in \mathcal{G}l(d)$ is such that $\lambda(V^*V) = \gamma$ then we have that*

$$\sum_{i \in \mathbb{I}_d} h\big(\lambda_i(S_{\mathcal{F}})\gamma_{d+1-i}\big) \leq P_h(V \cdot \mathcal{F}), \tag{34}$$

and this lower bound is attained.

2. *If we assume further that $h \in \mathrm{Conv}_s(\mathbb{R}_{\geq 0})$ then equality holds in Eq. (34) iff there exists an o.n.b. $\{v_i\}_{i \in \mathbb{I}_d}$ such that*

$$S_{\mathcal{F}} = \sum_{i \in \mathbb{I}_d} \lambda_i(S_{\mathcal{F}}) v_i \otimes v_i \quad \textit{and} \quad V^*V = \sum_{i \in \mathbb{I}_d} \gamma_{d+1-i} v_i \otimes v_i. \tag{35}$$

Proof. Recall that if V and $\mathcal{F}$ are as above then $S_{V \cdot \mathcal{F}} = V S_{\mathcal{F}} V^*$. Hence, by Theorem 5.1 and the remarks in Section 2.3 we conclude that

$$\lambda(S_{\mathcal{F}}) \circ \lambda\big(V^*V\big)^{\uparrow} \prec_{\log} \lambda(S_{V \cdot \mathcal{F}}) \quad \Longrightarrow \quad \lambda(S_{\mathcal{F}}) \circ \lambda\big(V^*V\big)^{\uparrow} \prec_w \lambda(S_{V \cdot \mathcal{F}}).$$

The submajorization above together with the fact that h is an increasing and convex function imply Eq. (34). On the other hand, it is clear that this lower bound is attained. Assume now that the lower bound in Eq. (34) is attained for some V as above. Using the fact that h is (an increasing) strictly convex function and the submajorization relation above then we conclude that $(\lambda(S_{\mathcal{F}}) \circ \lambda(V^*V)^{\uparrow})^{\downarrow} = \lambda(S_{V \cdot \mathcal{F}})$ (see the remarks in Section 2.3). Thus, again by Theorem 5.1, we conclude that there exists an o.n.b. $\{v_i\}_{i \in \mathbb{I}_d}$ for $\mathbb{C}^d$ for which Eq. (35) holds. □

Notation 5.5. Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be a frame for $\mathbb{C}^d$ with frame operator $S_{\mathcal{F}} \in \mathcal{G}l(d)^+$. Let $0 < \delta < 1$ and let $s > 0$ be such that $(1-\delta)^d \leq s \leq (1+\delta)^d$. Then, with the notation from Eqs. (6), (7) and (21) then, the identity in Eq. (8) becomes

$$\mathcal{S}\big(\mathcal{P}_{(s,\delta)}(\mathcal{F})\big) = \mathcal{O}_{(s,\delta)}(S_{\mathcal{F}}).$$

Thus, the following result is a direct consequence of the previous identity and Theorem 5.2.

Theorem 5.6. *Let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be a frame for $\mathbb{C}^d$ with frame operator $S_{\mathcal{F}} \in \mathcal{G}l(d)^+$. Let δ and s as in Notation 5.5. Define the following data: $\lambda = \log \lambda(S_{\mathcal{F}}) = (\log \lambda_i(S_{\mathcal{F}}))_{i \in \mathbb{I}_d} \in (\mathbb{R}^d)^{\downarrow}$,*

$$t = \log\left(\frac{s^2 \cdot \det(S)}{(1-\delta)^{2d}}\right) \geq \mathrm{tr}(\lambda) \quad \textit{and} \quad \varepsilon = 2\log\left(\frac{1+\delta}{1-\delta}\right) > 0.$$

Let $\rho = \rho_{(t,\varepsilon)}(\lambda, 0) \in \Lambda_{(t,\varepsilon)}(\lambda, 0)$ be as in Theorem 4.3. Denote by $\mu = \mu_{(s,\delta)}(\mathcal{F})$ the vector

$$\mu = (1-\delta)^2 \cdot \exp \rho = \big((1-\delta)^2 e^{\rho_i}\big)_{i \in \mathbb{I}_d} \in \big(\mathbb{R}^d_{>0}\big)^{\downarrow}.$$

Then,

1. *There exists* $V_0 \cdot \mathcal{F} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$ *such that* $\lambda(S_{V_0 \cdot \mathcal{F}}) = \mu$.
2. *For every* $V \cdot \mathcal{F} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$ *we have that*

$$\prod_{i=1}^{k} \mu_i \leq \prod_{i=1}^{k} \lambda_i(S_{V \cdot \mathcal{F}}) \quad \textit{for every } k \in \mathbb{I}_d. \tag{36}$$

3. *Given* $V \cdot \mathcal{F} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$, *equality holds in Eq. (36) for every* $k \in \mathbb{I}_d$ *(i.e.* $\lambda(S_{V \cdot \mathcal{F}}) = \mu$*)* $\iff |\det(V)| = s$ *and there exists an o.n.b.* $\{v_i\}_{i \in \mathbb{I}_d}$ *for* $\mathbb{C}^d$ *such that*

$$S_{\mathcal{F}} = \sum_{i \in \mathbb{I}_d} \lambda_i(S_{\mathcal{F}}) v_i \otimes v_i \quad \textit{and} \quad V^* V = \sum_{i \in \mathbb{I}_d} \frac{\mu_i}{\lambda_i(S)} v_i \otimes v_i. \qquad \square \tag{37}$$

The previous result establishes the existence of perturbations $V \cdot \mathcal{F} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$ by equivalent frames, that are optimal in a rather structural sense. For example, these optimal perturbations are minimizers of convex potentials. In turn, these convex potential can be used to obtain a direct and simple (scalar) quantitative measure of performance of arbitrary perturbations within $\mathcal{P}_{(s,\delta)}(\mathcal{F})$. We formalize these remarks in the following:

Corollary 5.7. *Fix an increasing function* $h \in \operatorname{Conv}(\mathbb{R}_{\geq 0})$. *With the notation and terminology of Theorem 5.6:*

1. *For every* $V \cdot \mathcal{F} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$ *then*

$$\sum_{i \in \mathbb{I}_d} h(\mu_i) \leq P_h(V \cdot \mathcal{F}), \tag{38}$$

   and this lower bound is attained.
2. *Assume further that* $h \in \operatorname{Conv}_s(\mathbb{R}_{\geq 0})$. *Then,* $V \cdot \mathcal{F} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$ *attains the lower bound in (38) iff* $|\det(V)| = s$ *and there exists an o.n.b.* $\{v_i\}_{i \in \mathbb{I}_d}$ *for* $\mathbb{C}^d$ *such that Eq. (37) holds.*

Proof. With the notation above and using the fact that $S_{V \cdot \mathcal{F}} = V S_{\mathcal{F}} V^*$, then the remarks in Section 2.3 and Theorem 5.2 imply that

$$(\log \mu_i)_{i \in \mathbb{I}_d} \prec_w \big(\log\big(\lambda_i(S_{V \cdot \mathcal{F}})\big)\big)_{i \in \mathbb{I}_d} \quad \Longrightarrow \quad \mu \prec_w \lambda(S_{V \cdot \mathcal{F}}).$$

Then, the submajorization above together with the fact that h is an increasing and convex function imply Eq. (38). Assume further that the lower bound in Eq. (38) is attained for some V as above. Using the fact that h is (an increasing) strictly convex function and the submajorization relation above then we conclude that $\mu = \lambda(S_{V \cdot \mathcal{F}})$.

Hence, by Theorem 5.2, we conclude that there exists an o.n.b. $\{v_i\}_{i\in\mathbb{I}_d}$ for $\mathbb{C}^d$ for which Eq. (37) holds. $\square$

Remark 5.8. Consider the notation and terminology of Theorem 5.6. Let

$$V_0 = \sum_{i\in\mathbb{I}_d} \left(\frac{\mu_i}{\lambda_i(S)}\right)^{1/2} v_i \otimes v_i \in \mathcal{M}_d(\mathbb{C})^+.$$

1. Then, $V_0 \cdot \mathcal{F} = \{\tilde{f}_i\}_{i\in\mathbb{I}_n} \in \mathcal{P}_{(s,\delta)}(\mathcal{F})$ and $\lambda(S_{V_0\cdot\mathcal{F}}) = \mu$.
2. In this case, since $\|V_0 - I\| \leq \delta$, for every $j \in \mathbb{I}_n$ we have that

$$\|f_j - \tilde{f}_j\| = \big\|(T_\mathcal{F} - VT_\mathcal{F})(e_j)\big\| \leq \|V - I\|\|T_\mathcal{F}\| \leq \delta\|T_\mathcal{F}\|.$$

   In particular $\big|\|f_j\| - \|\tilde{f}_j\|\big| \leq \delta\|T_\mathcal{F}\|$, for $j \in \mathbb{I}_n$. $\triangle$

5.3. Expansive perturbations by equivalent frames

Arguing as in proof of Theorem 5.2 above, in terms of Theorem 5.1 and the results from [19] described in Remark 4.1, we can get the following perturbation result which is of independent interest.

Theorem 5.9. *Let $S \in \mathcal{G}l(d)^+$ and let $s > 1$. Define*

$$\lambda = \big(\log\lambda_i(S)\big)_{i\in\mathbb{I}_d} \in \big(\mathbb{R}^d\big)^\downarrow \quad \textit{and} \quad t = 2\log s + \operatorname{tr}\lambda > \operatorname{tr}\lambda.$$

Let $\nu = \nu(\lambda, t) \in \Lambda_t(\lambda)$ be as in Remark 4.1. Denote by $\mu = (e^{\nu_i})_{i\in\mathbb{I}_d}$. Then,

1. *There exists $V_0 \in \mathcal{G}l(d)$ expansive (i.e. $V_0^*V_0 \geq I$) such that*

$$|\det V_0| = s \quad \textit{and} \quad \lambda\big(V_0SV_0^*\big) = \mu.$$

2. *If $V \in \mathcal{G}l(d)$ is expansive and such that $|\det V| = s$, then $\mu \prec_{\log} \lambda(VSV^*)$.*
3. *Equalities hold in item 2 $\Longleftrightarrow$ there exists an o.n.b. (of eigenvectors) $\{v_i\}_{i\in\mathbb{I}_d}$ for $\mathbb{C}^d$ such that S and V^*V satisfy Eq. (26).* $\square$

We shall consider expansive perturbations of a fixed frame $\mathcal{F}$ by equivalent frames, with a fixed determinant parameter: Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ be a frame for $\mathbb{C}^d$ and let $s > 1$. Denote by

$$\mathcal{PE}_s(\mathcal{F}) = \big\{V\cdot\mathcal{F} : V \in \mathcal{G}l(d),\ |\det V| = s \text{ and } V^*V \geq I\big\}.$$

Note that, in contrast with $\mathcal{P}_{(s,\delta)}(\mathcal{F})$, the sets $\mathcal{PE}_s(\mathcal{F})$ are closed by unitary equivalence.

Corollary 5.10. *Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ be a frame for $\mathbb{C}^d$ with frame operator $S_{\mathcal{F}} \in \mathcal{G}l(d)^+$. Let $s > 1$. Consider the data λ, t, ν and μ as in Theorem 5.9 with $S = S_{\mathcal{F}}$. Then,*

1. *There exists $V_0 \cdot \mathcal{F} \in \mathcal{PE}_s(\mathcal{F})$ such that $\lambda(S_{V_0\cdot\mathcal{F}}) = \mu$.*
2. *For every other $V \cdot \mathcal{F} \in \mathcal{PE}_s(\mathcal{F})$ we have that $\prod_{i=1}^k \mu_i \le \prod_{i=1}^k \lambda_i(S_{V\cdot\mathcal{F}})$ for $k \in \mathbb{I}_d$.*
3. *Equality holds in item 2 for every $k \in \mathbb{I}_d$ (i.e. $\lambda(S_{V\cdot\mathcal{F}}) = \mu$) $\Longleftrightarrow$ there exists an o.n.b. $\{v_i\}_{i\in\mathbb{I}_d}$ for $\mathbb{C}^d$ such that $S_{\mathcal{F}}$ and V^*V satisfy Eq. (37).* □

Appendix A. Multiplicative Lidskii's inequality

Although majorization and log-majorization are not total relations in $\mathbb{R}^d$ they appear naturally in many situations in matrix analysis. Some examples of this phenomenon are the Schur–Horn theorem characterizing the main diagonals of unitary conjugates of a selfadjoint matrix A, and Horn's relations between the eigenvalues and singular values of matrices. Another interesting example of a majorization relation is Lidskii's inequality for selfadjoint matrices; namely, if A, B are selfadjoint matrices then $\lambda(A) + \lambda^{\uparrow}(B) \prec \lambda(A + B)$. Lidskii's inequality has a multiplicative version obtained by Li and Mathias in [14]. In the positive invertible case, Li and Mathias's results can be put into the deep theory of singular value inequalities developed by Klyachko in [12]. Next, we describe these results in detail and characterize the case of equality in Li–Mathias's multiplicative inequality. The following inequalities are the multiplicative version of Lidskii's inequality:

Theorem A.1. *(See [14].) Let $S \in \mathcal{M}_d(\mathbb{C})^+$ and $V \in \mathcal{G}l(d)$. Let $J \subseteq \mathbb{I}_d$ be such that $|J| = k$ and $\lambda_i(S) > 0$ for $i \in J$. Then we have that*

$$\prod_{i=1}^k \lambda_{d+1-i}(V^*V) \le \prod_{i\in J} \frac{\lambda_i(VSV^*)}{\lambda_i(S)} \le \prod_{i=1}^k \lambda_i(V^*V). \qquad \square$$

We complement this result by characterizing the case of equality in the inequalities above. We shall use some results of [20, Section 8], where we study the case of equality in the additive Lidskii's inequalities, and also some combinatorial problems. We begin by revisiting the following well known inequality from matrix theory. Our interest relies in the case of equality.

Lemma A.2 *(Weyl's inequalities). Let $A, B \in \mathcal{M}_d(\mathbb{C})$ be Hermitian matrices. Then,*

$$\lambda_i(A + B) \le \lambda_i(A) + \lambda_1(B) \quad \text{for every } i \in \mathbb{I}_d. \tag{39}$$

If there is $i \in \mathbb{I}_d$ such that $\lambda_i(A + B) = \lambda_i(A) + \lambda_1(B)$ then there is a unit vector x such that

$$Ax = \lambda_i(A)x \quad \text{and} \quad Bx = \lambda_1(B)x.$$

Proof. It is a particular case of [20, Lemma 8.1]. $\square$

Proposition A.3 *(Ostrowski's inequality). Let $S \in \mathcal{M}_d(\mathbb{C})^+$ and let $V \in \mathcal{M}_d(\mathbb{C})$ be such that $V^*V \geq I$. Then, for $i \in \mathbb{I}_d$ we have that*

$$\lambda_i(S) \leq \lambda_i\big(VSV^*\big). \tag{40}$$

Moreover, if there exists $J \subset \mathbb{I}_d$ such that $\lambda_i(S) = \lambda_i(VSV^*)$ for $i \in J$, then there exists an o.n.s. $\{v_i\}_{i\in I} \subset \mathbb{C}^d$ such that

$$|V|v_i = v_i \quad \textit{and} \quad Sv_i = \lambda_i(S)v_i \quad \textit{for } i \in I.$$

Proof. The first part of the statement is well known (see for example [6, Theorem 5.4.9]). Hence, we prove the second part of the statement by induction on $|J|$, the number of elements of J. Assume first that $V \in \mathcal{M}_d(\mathbb{C})^+$ is such that $V^2 \geq I$ i.e. that V is a positive expansion. Fix $i \in J$ and notice that, by Sylvester's law of inertia, $\lambda_i(V(S - \lambda_i(S)I)V) = 0$, since $\lambda_i(S - \lambda_i(S)I) = 0$. By Lemma A.2 we have that

$$\begin{aligned} \lambda_i(VSV) - \lambda_i(S)\lambda_d\big(V^2\big) &= \lambda_i(VSV) + \lambda_1\big(-\lambda_i(S)V^2\big) \\ &\overset{\text{A.2}}{\geq} \lambda_i\big(V\big(S - \lambda_i(S)I\big)V\big) = 0. \end{aligned} \tag{41}$$

Since $\lambda_d(V^2) \geq 1$ and $\lambda_i(VSV) = \lambda_i(S)$ $(i \in J)$ we conclude that $\lambda_d(V^2) = 1$. Moreover, by the equality in Eq. (41) and Lemma A.2, there is $x \in \mathbb{C}^d$, $\|x\| = 1$ such that $VSVx \overset{\text{A.2}}{=} \lambda_i(VSV)x$ and

$$-\lambda_i(S)V^2x \overset{\text{A.2}}{=} \lambda_1\big(-\lambda_i(S)V^2\big)x = -\lambda_i(S)\lambda_d\big(V^2\big)x = -\lambda_i(S)x.$$

Hence $V^2x = x$ and then $Vx = x$. Thus, $VSVx = \lambda_i(VSV)x \Longrightarrow Sx = \lambda_i(VSV)x = \lambda_i(S)x$.

This proves the statement for $|J| = 1$. If we assume that $|J| > 1$ then we fix $v_i = x$ and consider $\mathcal{W} = \{v_i\}^\perp$, which reduces both A and V. Therefore an easy inductive argument involving the restriction $S|_\mathcal{W}$ and $V|_\mathcal{W}$ shows the general case.

If we now consider an arbitrary $V \in \mathcal{M}_d(\mathbb{C})$ such that $V^*V \geq I$, then let $V = U|V|$ be the polar decomposition of V. In this case, U is a unitary operator and $VSV^* = U|V|S|V|U^*$ so that $\lambda_i(VSV^*) = \lambda_i(|V|S|V|)$ for every $i \in \mathbb{I}_d$ and $|V|^2 = V^*V \geq I$. These last facts together with the case of equality for positive expansions prove the statement. $\square$

In order to state our results we introduce the following notion.

Definition A.4. Let $S \in \mathcal{G}l(d)^+$ and let $V \in \mathcal{G}l(d)$. We say that V is an upper multiplicative matching (UMM) of S (resp. lower MM or LMM of S) if there exists a family $\{J_k\}_{k\in\mathbb{I}_d}$ such that $J_k \subseteq J_{k+1} \subseteq \mathbb{I}_d$ for $1 \leq k \leq d-1$, $|J_k| = k$ for $k \in \mathbb{I}_d$ and such that

$$\prod_{i\in J_k} \frac{\lambda_i(VSV^*)}{\lambda_i(S)} = \prod_{i=1}^{k} \lambda_i\big(V^*V\big), \quad k \in \mathbb{I}_d$$

(resp. $\prod_{i=1}^{k} \lambda_i^{\uparrow}(V^*V) = \prod_{i=1}^{k} \lambda_{d+1-i}(V^*V) = \prod_{i\in J_k} \frac{\lambda_i(VSV^*)}{\lambda_i(S)}$, $k \in \mathbb{I}_d$). △

Theorem A.5. *Let $S \in \mathcal{G}l(d)^+$ and let $V \in \mathcal{G}l(d)$ be an UMM or an LMM of S. Then S and $|V|$ commute.*

Proof. We can assume that V is not a multiple of the identity. We use the splitting technique considered in [14]. Let V be an UMM of S and assume further that $V \in \mathcal{G}l(d)^+$. Denote by $\lambda = (\lambda_1, \dots, \lambda_d) = \lambda(V) \in (\mathbb{R}^d_{>0})^{\downarrow}$ and fix be an o.n.b. $\{u_i\}_{i\in\mathbb{I}_d}$ for $\mathbb{C}^d$ such that $V = \sum_{i\in\mathbb{I}_d} \lambda_i u_i \otimes u_i$. Fix an index $2 \le k \le d$ be such that $\lambda_{k-1} > \lambda_k$ and denote by $V_k = \lambda_k^{-1} V$, which is also an UMM for S. In this case $\lambda_i(V_k) = \frac{\lambda_i}{\lambda_k}$ for every $i \in \mathbb{I}_d$. In particular $\lambda_{k-1}(V_k) > \lambda_k(V_k) = 1$. Let

$$\mu = \big(\lambda_1(V_k), \dots, \lambda_{k-1}(V_k), \mathbb{1}_{d-k}\big) \in \big(\mathbb{R}^d_{>0}\big)^{\downarrow} \quad \text{and} \quad B_k = \sum_{i\in\mathbb{I}_d} \mu_i u_i \otimes u_i \in \mathcal{G}l(d)^+.$$

Notice that $W_k = \ker(B_k - I) = \operatorname{span}\{u_i : k \le i \le d\} \Longrightarrow \dim W_k = d + 1 - k$. Also notice that the orthogonal projection $P_k \overset{\text{def}}{=} P_{W_k}$ coincides with the spectral projection of V corresponding to the interval $(0, \lambda_k]$.

On the other hand, by construction of B_k, we see that $B_k \ge I$ and $V_k^{-1} B_k = B_k V_k^{-1} \ge I$. Using that V_k is an UMM of S, we can take $J_{k-1} \subseteq \mathbb{I}_d$ such that $|J_{k-1}| = k - 1$ and

$$\prod_{i\in J_{k-1}} \frac{\lambda_i(V_k S V_k)}{\lambda_i(S)} = \prod_{i=1}^{k-1} \lambda_i\big(V_k^2\big). \tag{42}$$

Then, by Ostrowski's inequality we get that

$$\prod_{i\in J_{k-1}} \frac{\lambda_i(V_k S V_k)}{\lambda_i(S)} \overset{(40)}{\le} \prod_{i\in J_{k-1}} \frac{\lambda_i((B_k V_k^{-1})(V_k S V_k)(V_k^{-1} B_k))}{\lambda_i(S)} = \prod_{i\in J_{k-1}} \frac{\lambda_i(B_k S B_k)}{\lambda_i(S)}.$$

Using Ostrowski's inequality again, we see that $\frac{\lambda_i(B_k S B_k)}{\lambda_i(S)} \ge 1$ for every $i \in \mathbb{I}_d$ and therefore

$$\begin{aligned}\prod_{i\in J_{k-1}} \frac{\lambda_i(V_k S V_k)}{\lambda_i(S)} &\le \prod_{i\in J_{k-1}} \frac{\lambda_i(B_k S B_k)}{\lambda_i(S)} \le \prod_{i\in\mathbb{I}_d} \frac{\lambda_i(B_k S B_k)}{\lambda_i(S)} \\ &= \frac{\det(B_k S B_k)}{\det(S)} = \det\big(B_k^2\big) = \prod_{i=1}^{k-1} \lambda_i\big(V_k^2\big).\end{aligned}$$

By Eq. (42) we see that the previous inequalities are actually equalities. Hence, if we let $J_{k-1}^c = \mathbb{I}_d \setminus J_{k-1}$ then $|J_{k-1}^c| = d+1-k$ and

$$\prod_{i\in J_{k-1}^c} \frac{\lambda_i(B_k S B_k)}{\lambda_i(S)} = 1 \quad \Longrightarrow \quad \lambda_i(B_k S B_k) = \lambda_i(S) \quad \text{for } i \in J_{k-1}^c.$$

By the case of equality in Ostrowski's inequality in Proposition A.3 we conclude that there exists an o.n.s. $\{v_i\}_{i\in J_{k-1}^c} \subseteq \mathbb{C}^d$ such that

$$B_k v_i = v_i \quad \text{and} \quad S v_i = \lambda_i(S) \quad \text{for } i \in J_{k-1}^c. \tag{43}$$

Then we conclude that $\{v_i\}_{i\in J_{k-1}^c}$ is another o.n.b. of W_k. Hence $P_k = \sum_{i\in J_{k-1}^c} v_i \otimes v_i$ and, by Eq. (43), we conclude that P_k and S commute. Finally, since V can be written as a linear combination of its spectral projections P_k (for $\lambda_{k-1} > \lambda_k$) and the identity I, we see that V and S commute in this case. The general case for arbitrary $V \in \mathcal{G}l(d)$ follows from the positive case with the reduction described at the end of the proof of Proposition A.3.

Assume now that $V \in \mathcal{G}l(d)^+$ is a LMM of S. Then V^{-1} is an UMM for VSV. Indeed, if $J_k \subseteq \mathbb{I}_d$ is such that

$$\prod_{i=1}^k \lambda_i^\uparrow(V^2) = \prod_{i\in J_k} \frac{\lambda_i(VSV)}{\lambda_i(S)},$$

then we have that

$$\prod_{i=1}^k \lambda_i(V^{-2}) = \left(\prod_{i=1}^k \lambda_i^\uparrow(V^2)\right)^{-1} = \left(\prod_{i\in J_k} \frac{\lambda_i(VSV)}{\lambda_i(S)}\right)^{-1} = \prod_{i\in J_k} \frac{\lambda_i(V^{-1}(VSV)V^{-1})}{\lambda_i(VSV)}.$$

By the first part of this proof, we conclude that V^{-1} and VSV commute, which implies that S and V commute. If $V \in \mathcal{G}l(d)$ is arbitrary we conclude that S and $|V|$ commute, using the reduction to the positive case described at the end of the proof of Proposition A.3. $\square$

Now we can re-state and prove Theorem 5.1:

Theorem 5.1. *Let $S \in \mathcal{G}l(d)^+$ and let $\gamma \in (\mathbb{R}_{>0}^d)^\downarrow$. Then, for every $V \in \mathcal{G}l(d)$ such that $\lambda(V^*V) = \gamma$ we have that*

$$\lambda(S) \circ \gamma^\uparrow \prec_{\log} \lambda(VSV^*) \prec_{\log} \lambda(S) \circ \gamma \in (\mathbb{R}_{>0}^d)^\downarrow.$$

Moreover, if $\lambda(VSV^*) = (\lambda(S) \circ \gamma^\uparrow)^\downarrow$ (resp. $\lambda(VSV^*) = \lambda(S) \circ \gamma$) then there exists an o.n.b. $\{v_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that

$$S = \sum_{i \in \mathbb{I}_d} \lambda_i(S) v_i \otimes v_i \quad \textit{and} \quad |V| = \sum_{i \in \mathbb{I}_d} \gamma_{d+1-i}^{1/2} v_i \otimes v_i \tag{44}$$

(resp. $S = \sum_{i \in \mathbb{I}_d} \lambda_i(S) v_i \otimes v_i$ *and* $|V| = \sum_{i \in \mathbb{I}_d} \gamma_i^{1/2} v_i \otimes v_i$*).*

Proof. Let S and V be as above. Assume further that $V \in \mathcal{G}l(d)^+$ and notice that then $\lambda(VSV) = \lambda(S^{1/2}V^2S^{1/2})$. By Theorem A.1 we get that, for every $J \subset \mathbb{I}_d$ with $|J| = k$,

$$\prod_{i \in J} \lambda_i^\uparrow(S)\lambda_i(V^2) = \prod_{i \in J} \frac{\lambda_i(S^{-1/2}(S^{1/2}V^2S^{1/2})S^{-1/2})}{\lambda_i(S^{-1})} \le \prod_{i=1}^k \lambda_i(S^{1/2}V^2S^{1/2}).$$

This shows that $\lambda \circ \lambda^\uparrow(S) \prec_{\log} \lambda(VSV)$ or equivalently, that $\lambda(S) \circ \lambda^\uparrow \prec_{\log} \lambda(VSV)$. Moreover, the previous facts also show that if $\lambda(VSV) = (\lambda(S) \circ \lambda^\uparrow)^\downarrow$ then $S^{-1/2}$ is an UMM of $S^{1/2}V^2S^{1/2}$. By Theorem A.5 we see that $S^{-1/2}$ and $S^{1/2}V^2S^{1/2}$ commute, which in turn implies that S and V commute.

Since S and V commute we conclude that there exists an o.n.b. $\{w_i\}_{i \in \mathbb{I}_d}$ of $\mathbb{C}^d$ such that

$$S = \sum_{i \in \mathbb{I}_d} \lambda_i(S) w_i \otimes w_i \quad \text{and} \quad V = \sum_{i \in \mathbb{I}_d} \lambda_{\sigma(i)}^\uparrow(V) w_i \otimes w_i$$

for some permutation $\sigma \in \mathbb{S}_d$. That is, in this case we have that

$$\big(\lambda(S) \circ \lambda^\uparrow(V^2)\big)^\downarrow = \lambda(VSV) = \big(\lambda(S) \circ \lambda_\sigma^\uparrow(V^2)\big)^\downarrow.$$

Notice that by replacing S and V by tS and tV for sufficiently large $t > 0$ we can always assume that $S - I \in \mathcal{G}l(d)^+$ and $V - I \in \mathcal{G}l(d)^+$. Using the properties of the logarithm, we conclude that the vectors $\log \lambda(S)$ and $\log \lambda(V^2) \in (\mathbb{R}_{>0})^\downarrow$ are such that

$$\big(\log \lambda(S) + \log \lambda^\uparrow(V^2)\big)^\downarrow = \big(\log \lambda(S) + \log \lambda_\sigma^\uparrow(V^2)\big)^\downarrow.$$

By [20, Proposition 8.6 and Remark 8.7] we conclude that $\log \lambda(S) = \log \lambda_\sigma(S)$. That is, if we set $v_i = w_{\sigma^{-1}(i)}$ for $i \in \mathbb{I}_d$ then the o.n.b. $\{v_i\}_{i \in \mathbb{I}_d}$ satisfies the conditions in Eq. (44). The general case, for $V \in \mathcal{G}l(d)$, follows by the reduction described at the end of the proof of Proposition A.3.

On the other hand, notice that a direct application of Theorem A.1 shows that

$$\prod_{i=1}^k \frac{\lambda_i(VSV^*)}{\lambda_i(S)} \le \prod_{i=1}^k \lambda_i(V^*V) \quad \Longrightarrow \quad \prod_{i=1}^k \lambda_i(VSV^*) \le \prod_{i=1}^k \lambda_i(S)\lambda_i(V^*V).$$

Hence, we conclude that $\lambda(VSV^*) \prec_{\log} \lambda(S) \circ \lambda(V^*V) \in (\mathbb{R}_{>0}^d)^\downarrow$. Finally, in case that $\lambda(VSV^*) = \lambda(S) \circ \lambda(V^*V)$ we see that S is an UMM for S and therefore S and $|V|$ commute. In this case it is straightforward to check that there exists an o.n.b. $\{v_i\}_{i \in \mathbb{I}_d}$ with the desired properties. $\square$

References

[1] J. Antezana, P. Massey, M. Ruiz, D. Stojanoff, The Schur–Horn theorem for operators and frames with prescribed norms and frame operator, Illinois J. Math. 51 (2007) 537–560.
[2] R. Balan, Equivalence relations and distances between Hilbert frames, Proc. Amer. Math. Soc. 127 (8) (1999) 2353–2366.
[3] R. Bhatia, Matrix Analysis, Springer, Berlin, Heidelberg, New York, 1997.
[4] J. Cahill, M. Fickus, D.G. Mixon, M.J. Poteet, N. Strawn, Constructing finite frames of a given spectrum and set of lengths, Appl. Comput. Harmon. Anal. 35 (1) (2013) 52–73.
[5] J.J. Benedetto, M. Fickus, Finite normalized tight frames, Adv. Comput. Math. 18 (2–4) (2003) 357–385.
[6] R.A. Horn, C.R. Johnson, Matrix Analysis, second edition, Cambridge University Press, Cambridge, 2013.
[7] P.G. Casazza, M. Fickus, J. Kovacevic, M.T. Leon, J.C. Tremain, A physical interpretation of tight frames, in: Harmonic Analysis and Applications, in: Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2006, pp. 51–76.
[8] P.G. Casazza, G. Kutyniok (Eds.), Finite Frames: Theory and Applications, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, 2013.
[9] O. Christensen, An Introduction to Frames and Riesz Bases, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2003.
[10] M. Fickus, J. Marks, M.J. Poteet, A generalized Schur–Horn theorem and optimal frame completions, preprint, http://arxiv.org/abs/1408.2882.
[11] M. Fickus, D.G. Mixon, M.J. Poteet, Frame completions for optimally robust reconstruction, in: Proceedings of SPIE, vol. 8138, 2011, 81380Q/1–8.
[12] A.A. Klyachko, Random walks on symmetric spaces and inequalities for matrix spectra, in: Special Issue: Workshop on Geometric and Combinatorial Methods in the Hermitian Sum Spectral Problem, Coimbra, 1999, Linear Algebra Appl. 319 (1–3) (2000) 37–59.
[13] J. Leng, D. Han, Optimal dual frames for erasures II, Linear Algebra Appl. 435 (2011) 1464–1472.
[14] C.K. Li, R. Mathias, The Lidskii–Mirsky–Wielandt theorem – additive and multiplicative versions, Numer. Math. 81 (3) (1999) 377–413.
[15] J. Lopez, D. Han, Optimal dual frames for erasures, Linear Algebra Appl. 432 (2010) 471–482.
[16] P. Massey, M. Ruiz, Minimization of convex functionals over frame operators, Adv. Comput. Math. 32 (2010) 131–153.
[17] P. Massey, M. Ruiz, D. Stojanoff, Duality in reconstruction systems, Linear Algebra Appl. 436 (2012) 447–464.
[18] P. Massey, M. Ruiz, D. Stojanoff, Robust dual reconstruction systems and fusion frames, Acta Appl. Math. 119 (2012) 167–183.
[19] P. Massey, M. Ruiz, D. Stojanoff, Optimal dual frames and frame completions for majorization, Appl. Comput. Harmon. Anal. 34 (2) (2013) 201–223.
[20] P. Massey, M. Ruiz, D. Stojanoff, Optimal frame completions, Adv. Comput. Math. (2014), http://dx.doi.org/10.1007/s10444-013-9339-7, in press.
[21] P. Massey, M. Ruiz, D. Stojanoff, Optimal frame completions with prescribed norms for majorization, J. Fourier Anal. Appl. 20 (5) (2014) 1111–1140.